OXFORD IB DIPLOMA PROGRAMME

WORKED SOLUTIONS

MATHEMATICS HIGHER LEVEL: CALCULUS

COURSE COMPANION

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OXFORD

1 Patterns to infinity

Skills check

1 a $\dfrac{3}{4n+1} - \dfrac{3}{4n-1} = \dfrac{3(4n-1)-3(4n+1)}{(4n+1)(4n-1)}$

$= \dfrac{12n-3-12n-3}{16n^2-1} = \dfrac{-6}{16n^2-1}$

b $\dfrac{n-1}{n+2} - \dfrac{n}{n+3} = \dfrac{(n-1)(n-3)-n(n+2)}{(n+2)(n+3)}$

$= \dfrac{n^2-n+3n-3-n^2-2n}{n^2+2n+3n+6}$

$= -\dfrac{3}{n^2+5n+6}$

2 a $|3x-1| < 1 \Rightarrow -1 < 3x-1 < 1$

$\Rightarrow 0 < 3x < 2 \Rightarrow 0 \le x \le \dfrac{2}{3}$

b $|2x-3| \ge 4 \Rightarrow 2x-3 \ge 4$ or $2x-3 \le -4$

$\Rightarrow 2x \ge 7$ or $2x \le -1$

$\Rightarrow x \ge \dfrac{7}{2}$ or $x \le -\dfrac{1}{2}$

3 a $\dfrac{2n}{1+\sqrt{n}} = \dfrac{2n(1-\sqrt{n})}{(1+\sqrt{n})(1-\sqrt{n})} = \dfrac{2n(\sqrt{n}-1)}{n-1}$

b $\dfrac{n-1}{\sqrt{n}+\sqrt{n+1}} = \dfrac{(n-1)(\sqrt{n}-\sqrt{n+1})}{(\sqrt{n}+\sqrt{n+1})(\sqrt{n}-\sqrt{n+1})}$

$= \dfrac{(n-1)(\sqrt{n}-\sqrt{n+1})}{n-n-1}$

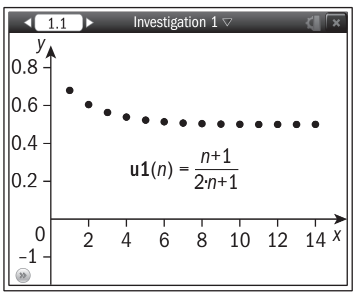
$= (n-1)(\sqrt{n+1}-\sqrt{n})$

4 a $\dfrac{(-1)^{n+1}}{2^n}$ **b)** $\dfrac{(-1)^{n+1}}{n+1}$ **c)** $\dfrac{2n-1}{2n}$

d $(-1)^{n+1}\sqrt{\dfrac{2n+1}{2^n}},\ n \in \mathbb{Z}^+$

Investigation 1

1



2 As n increases the terms get very close to $\dfrac{1}{2}$.

3 $\left|\dfrac{n+1}{2n+1} - \dfrac{1}{2}\right| < \dfrac{1}{10} \Rightarrow \left|\dfrac{2n+2-2n-1}{2(2n+1)}\right| < \dfrac{1}{10}$

$\Rightarrow \dfrac{1}{2n+1} < \dfrac{1}{5}$

$\Rightarrow n > 2 \therefore m = 3$

4 $\left|\dfrac{n+1}{2n+1} - \dfrac{1}{2}\right| < \varepsilon \Rightarrow \left|\dfrac{2n+2-2n-1}{2(2n+1)}\right| < \varepsilon$

$\Rightarrow \dfrac{1}{2n+1} < 2\varepsilon \Rightarrow n > \dfrac{1}{2}\left(\dfrac{1}{2\varepsilon} - 1\right)$

$\therefore \varepsilon = 0.01 \Rightarrow m = 25$; $\varepsilon = 0.001 \Rightarrow m = 250$;
$\varepsilon = 0.0001 \Rightarrow m = 2500$

5 $\left|\dfrac{n+1}{2n+1} - \underbrace{\dfrac{2}{5}}_{0.4}\right| < \dfrac{1}{10} \Rightarrow \left|\dfrac{2n+2-4n-2}{2(2n+1)}\right| < \dfrac{1}{10}$

$\Rightarrow \dfrac{2n}{2n+1} < \dfrac{1}{5}$

$\Rightarrow n < \dfrac{1}{8}$

No as this implies that $n < \dfrac{1}{8}$.

6 As n increases the terms get very close to 0.

7 $\left|\left(-\dfrac{1}{3}\right)^n\right| < \varepsilon \Rightarrow \dfrac{1}{3^n} < \varepsilon \Rightarrow n > \log_3\left(\dfrac{1}{\varepsilon}\right)$

$\therefore \varepsilon = 0.01 \Rightarrow m = 6$; $\varepsilon = 0.001 \Rightarrow m = 7$;
$\varepsilon = 0.0001 \Rightarrow m = 9$

8 $|u_n - L|$ can take arbitrary small values as we consider just terms after an order m (as we increase this order the terms are closer to L).

Exercise 1A

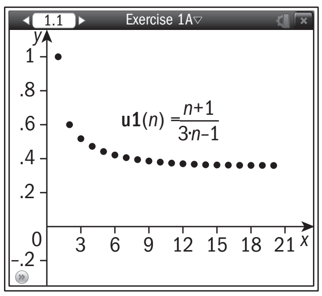
1 $\left|\dfrac{n+3}{2n+1} - \dfrac{1}{2}\right| < \dfrac{1}{1000} \Rightarrow \left|\dfrac{2n+6-2n-1}{2(2n+1)}\right| < \dfrac{1}{1000}$

$\Rightarrow \dfrac{5}{2n+1} < \dfrac{1}{500}$

$\Rightarrow n > 1249.5$

$\therefore m = 1250$

2 a



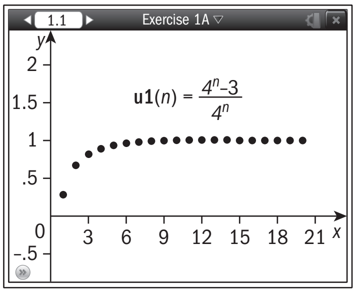
b $\left|\frac{n+1}{3n-1}-\frac{1}{3}\right| < \frac{1}{1000} \Rightarrow \left|\frac{3n+3-3n+1}{3(3n-1)}\right| < \frac{1}{1000}$

$\Rightarrow \frac{4}{9n-3} < \frac{1}{1000}$

$\Rightarrow n > \frac{4003}{9}$

$\therefore m = 445$

3 a



b $\left|\frac{4^n-3}{4^n}-1\right| < \frac{5}{10000} \Rightarrow \left|\frac{4^n-3-4^n}{4^n}\right| < \frac{1}{2000}$

$\Rightarrow \frac{3}{4^n} < \frac{1}{2000}$

$\Rightarrow n > \log_4 6000$

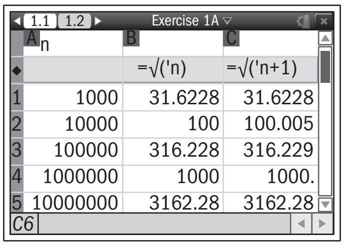
$\therefore m = 7$

4 a Divergent as the terms of even order approach $\frac{3}{4}$ and the terms of odd order approach $-\frac{3}{4}$.

b Convergent to zero as the absolute value of the terms get close to zero quickly.

c Divergent as while the terms of even order get close to zero, the terms of odd order increase without bound.

d Converges to zero as the square roots of consecutive integers are approximately equal for large values of n.


1.1 1.2 Exercise 1A

	A n	B	C
◆		=√('n)	=√('n+1)
1	1000	31.6228	31.6228
2	10000	100	100.005
3	100000	316.228	316.229
4	1000000	1000	1000.
5	10000000	3162.28	3162.28

C6

e Diverges to positive infinity as the terms increase without bound.

f Diverges as the terms follow the cyclic pattern $1, 0, -1, 0, 1, 0, -1, 0, \ldots$.

g Diverges as the terms of even order are equal to 1 and the terms of odd order get close to zero.

h Converges to $\frac{1}{2}$ as $\frac{n^3+n+1}{2n^3+3} = \frac{1}{2}\left(1+\frac{n-2}{2n^3+3}\right)$ and $\frac{n-2}{2n^3+3}$ approaches zero as n increases.

5 $\frac{n\cos^2(na)}{n^2+3} = \underbrace{\frac{n}{n^2+3}}_{\text{converges to zero}} \cdot \underbrace{\cos^2(na)}_{\text{bounded}}$ as $0 \le \cos^2(na) \le 1$.

Therefore $\lim_{n\to\infty}\frac{n\cos^2(na)}{n^2+3} = 0$, $a \in \mathbb{R}$. QED

Exercise 1B

1 a $-2\cdot\left(\frac{3}{5}\right)^n \le 2\cdot(-1)^{n-1}\cdot\left(\frac{3}{5}\right)^n \le 2\cdot\left(\frac{3}{5}\right)^n$; as

$\lim_{n\to\infty}\left(-2\cdot\left(\frac{3}{5}\right)^n\right) = \lim_{n\to\infty} 2\cdot\left(\frac{3}{5}\right)^n = 0$, by the

squeeze theorem $\lim_{n\to\infty} 2\cdot(-1)^{n-1}\cdot\left(\frac{3}{5}\right)^n$.

2 a $-1 \le \sin(2n) \le 1$, for all $n \in \mathbb{Z}^+$

$\therefore 3n-1 \le 3n+\sin(2n) \le 3n+1$

$\Rightarrow \frac{3n-1}{4n-3} \le \frac{3n+\sin(2n)}{4n-3} \le \frac{3n+1}{4n-3}$

as $4n-3 > 0$

for all $n \in \mathbb{Z}^+$; as $\lim_{n\to\infty}\frac{3n-1}{4n-3}$

b $\lim_{n\to\infty}\frac{3n-1}{4n-3} = \lim_{n\to\infty}\frac{3-\frac{1}{n}}{4-\frac{3}{n}} = \frac{3-0}{4-0} = \frac{3}{4}$ and

$\lim_{n\to\infty}\frac{3n+1}{4n-3} = \lim_{n\to\infty}\frac{3+\frac{1}{n}}{4-\frac{3}{n}} = \frac{3+0}{4-0} = \frac{3}{4}$

c As $\frac{3n-1}{4n-3} \le \frac{3n+\sin(2n)}{4n-3} \le \frac{3n+1}{4n-3}$ and

$\lim_{n\to\infty}\frac{3n-1}{4n-3} = \lim_{n\to\infty}\frac{3n+1}{4n-3} = \frac{3}{4}$ by the squeeze

theorem $\lim_{n\to\infty}\frac{3n+\sin(2n)}{4n-3} = \frac{3}{4}$

3 a $\frac{2n}{3n+1} \ge \frac{1}{2} \Rightarrow 4n \ge 3n+1 \Rightarrow n \ge 1$ which is true

for all $n \in \mathbb{Z}^+$; $\frac{2n}{3n+1} < \frac{2}{3} \Rightarrow 6n < 6n+2 \Rightarrow 0 < 2$

which is true for all $n \in \mathbb{Z}^+$.

Therefore, $\frac{1}{2} \le \frac{2n}{3n+1} < \frac{2}{3}$ for all $n \in \mathbb{Z}^+$.

b As $\frac{1}{2} \le \frac{2n}{3n+1} < \frac{2}{3}$ for all $n \in \mathbb{Z}^+$ and

$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = \lim_{n\to\infty}\left(\frac{3}{4}\right)^n = 0$ by the squeeze theorem,

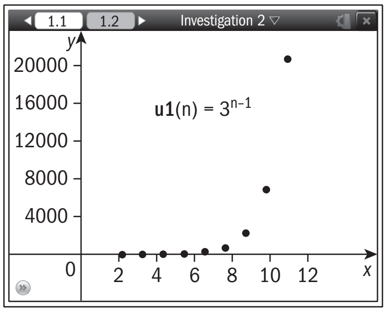
$\lim_{n\to\infty}\left(\frac{2n}{3n+1}\right)^n = 0$

4 Assumption: the sequence $\{a_n\}$ converges to a real number a. As $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = a$, then

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\frac{a_n+2}{3} \Rightarrow a = \frac{a+2}{3} \Rightarrow a = 1$

Investigation 2

1



2 The terms of the sequence increase without upper bound.

3 $3^{n-1} > 100 \Rightarrow n > 5.19... \Rightarrow m = 6$

4 $3^{n-1} \geq 1000 \Rightarrow n \geq 7.28... \Rightarrow m = 8,$

$3^{n-1} \geq 10000 \Rightarrow n \geq 9.38... \Rightarrow m = 10$ and

$3^{n-1} \geq 100000 \Rightarrow n \geq 11.47... \Rightarrow m = 12$

$\log_3(100)+1$	5.19181
$\log_3(1000)+1$	7.28771
$\log_3(10000)+1$	9.38361
$\log_3(100000)+1$	11.4795

4/99

5 $-2^n \leq 1000 \Rightarrow n > 9.96... \Rightarrow m = 10$,

$-2^n \leq 10000 \Rightarrow n > 13.2... \Rightarrow m = 14$ and

$-2^n \leq 1000000 \Rightarrow n > 16.6... \Rightarrow m = 17$

$\log_2(1000)$	9.96578
$\log_2(10000)$	13.2877
$\log_2(100000)$	16.6096

3/99

6 The terms of the sequence decrease without lower bound.

7

$\log_4(1000)$	4.98289
$\log_4(10000)$	6.64386
$\log_4(100000)$	8.30482

3/99

$|-4^n| \geq 1000 \Rightarrow n > 4.98... \Rightarrow m = 5,$

$|-4^n| \geq 10000 \Rightarrow n > 6.64... \Rightarrow m = 7$ and

$|-4^n| \geq 100000 \Rightarrow n > 8.30... \Rightarrow m = 9$

8 The absolute value of the terms of the sequence increase without upper bound.

Investigation 3

a

	A n	B	C
◆		='n/2^'n	
1	1.	0.5	
2	10.	0.00976563	
3	100.	7.8886091E-29	
4	500.	1.5274682E-148	
5	1000.	9.3326362E-299	

B5 =9.3326361850171E-299

$\therefore \lim_{n\to\infty} a_n = 0$

b Explore the limit for different values of k and b. For example:

	A n	B	C
◆		='n^2/3^('n)	='n^2/4^('n)
1	1.	0.33333333	0.25
2	10.	0.00169351	0.00009537
3	100.	1.9403252E-44	6.2230153E-57
4	500.	6.8756324E-23...	2.333159E-296
5	1000.	7.563891E-47...	8.7098098E-597

C5 =8.7098098162032E-597

	A n	B	C
◆		=('n)/3^('n)	='n/4^'n
1	1.	0.33333333	0.25
2	10.	0.00016935	0.00000954
3	100.	1.9403252E-46	6.2230153E-59
4	500.	1.3751265E-23...	4.6663181E-299
5	1000.	7.5638913E-47...	8.7098098E-600

C5 =8.7098098162032E-600

$\therefore \lim_{n\to\infty} b_n = 0$

c

	A n	B	C
◆		=root(3,'n)	
1	100	1.01105	
2	1000	1.0011	
3	10000	1.00011	
4	100000	1.00001	
5			

B4 =1.0000109861832

$\therefore \lim_{n\to\infty} c_n = 1$

d

	A n	B	C
◆		=root('n, 'n)	
1	100	1.00693	
2	1000	1.00092	
3	10000	1.00012	
4	100000	1.00001	
5	1000000	1.	

B5 =1.0000016118109

$\therefore \lim_{n\to\infty} d_n = 1$

e $\lim_{n\to\infty} e_n = +\infty$

You need to use a powerful tool to calculate this one, e.g., Mathematica or its free version Wolfram Alpha.

Exercise 1C

1 **a** $\lim\limits_{n\to\infty}\dfrac{5n^3-n}{2n^3+6n^2-1}=\lim\limits_{n\to\infty}\dfrac{5-\frac{1}{n^2}}{2+\frac{6}{n}-\frac{1}{n^3}}=\dfrac{5-0}{2+0-0}=\dfrac{5}{2}$

b $\lim\limits_{n\to\infty}\dfrac{n^2+5n}{n^3+2n+3}=\lim\limits_{n\to\infty}\dfrac{\frac{1}{n}+\frac{5}{n^2}}{1+\frac{2}{n^2}+\frac{3}{n^3}}=\dfrac{0+0}{1+0+0}=0$

c $\lim\limits_{n\to\infty}\dfrac{n^2+6n-1}{n+1}=\lim\limits_{n\to\infty}\dfrac{1+\frac{6}{n}-\frac{1}{n^2}}{\frac{1}{n}+\frac{1}{n^2}}=\dfrac{1+0-0}{0+0}=+\infty$

d $\lim\limits_{n\to\infty}(n^3-n)=\lim\limits_{n\to\infty}(n(n^2-1))=(+\infty)\times(+\infty)=+\infty$

e $\lim\limits_{n\to\infty}\left(\sqrt{n+5}-\sqrt{n+2}\right)$

$=\lim\limits_{n\to\infty}\dfrac{\left(\sqrt{n+5}-\sqrt{n+2}\right)\left(\sqrt{n+5}+\sqrt{n+2}\right)}{\sqrt{n+5}+\sqrt{n+2}}$

$=\lim\limits_{n\to\infty}\dfrac{3}{\sqrt{n+5}+\sqrt{n+2}}=\dfrac{3}{+\infty}=0$

f $\lim\limits_{n\to\infty}\sqrt{\dfrac{n+5}{n+2}}=\sqrt{\lim\limits_{n\to\infty}\dfrac{1+\frac{5}{n}}{1+\frac{2}{n}}}=\sqrt{\dfrac{1+0}{1+0}}=1$

g $\lim\limits_{n\to\infty}\dfrac{3^n+2^n}{4^n+5\cdot 3^n}=\lim\limits_{n\to\infty}\dfrac{1+\left(\frac{2}{3}\right)^n}{\left(\frac{4}{3}\right)^n+5}=\dfrac{1+0}{+\infty+5}=0$

h $\lim\limits_{n\to\infty}\dfrac{(-3)^{n+1}+7^n}{4^{n-1}+e^n}=\lim\limits_{n\to\infty}\dfrac{\left(-\frac{3}{7}\right)^n+1}{\frac{1}{4}\cdot\left(\frac{4}{7}\right)^n+\left(\frac{e}{7}\right)^n}=\dfrac{0+1}{0+0}=+\infty$

i $\lim\limits_{n\to\infty}\dfrac{\sqrt{n}-\sqrt[3]{n}}{\sqrt{n}}=\lim\limits_{n\to\infty}\left(1-\sqrt[6]{\dfrac{1}{n}}\right)=1-0=1$

j $\lim\limits_{n\to\infty}\left(\dfrac{n^3}{n+1}-\dfrac{n^3+1}{n}\right)=\lim\limits_{n\to\infty}\dfrac{n^4-n^4-n^3-n-1}{n^2+n}$

$=\lim\limits_{n\to\infty}\dfrac{-1-\frac{1}{n^2}-\frac{1}{n^3}}{\frac{1}{n}+\frac{1}{n^2}}$

$=\dfrac{-1-0-0}{0+0}=-\infty$

k $\lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=0}^{n}(2k-1)}{2n^2+1}=\lim\limits_{n\to\infty}\dfrac{\frac{2n-2}{2}\cdot n}{2n^2+1}=\lim\limits_{n\to\infty}\dfrac{n^2-n}{2n^2+1}$

$=\lim\limits_{n\to\infty}\dfrac{1-\frac{1}{n}}{2+\frac{1}{n^2}}=\dfrac{1-0}{2+0}=\dfrac{1}{2}$

l $\lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=0}^{n}(2^{k-1})}{3^3}=\lim\limits_{n\to\infty}\dfrac{\frac{1}{2}\,\frac{2^n-1}{2-1}}{3^n}=\lim\limits_{n\to\infty}\dfrac{2^n-1}{2\cdot 3^n}$

$=\lim\limits_{n\to\infty}\dfrac{\left(\frac{2}{3}\right)^n-\frac{1}{3^n}}{2}=\dfrac{0-0}{2}=0$

2 **a** $\lim\limits_{n\to\infty}\sqrt[n]{n}=1\Rightarrow\lim\limits_{n\to\infty}\ln(\sqrt[n]{n})=\ln 1=0$

$\Rightarrow\lim\limits_{n\to\infty}\dfrac{1}{n}\ln n=0\Rightarrow\lim\limits_{n\to\infty}\dfrac{\ln n}{n}=0$

As $\dfrac{\ln n}{n}\le\dfrac{\ln(n+k)}{n}\le\dfrac{\ln n+k}{n}$ and

$\lim\limits_{n\to\infty}\dfrac{\ln n}{n}=\lim\limits_{n\to\infty}\dfrac{\ln n+k}{n}=0$, by the squeeze

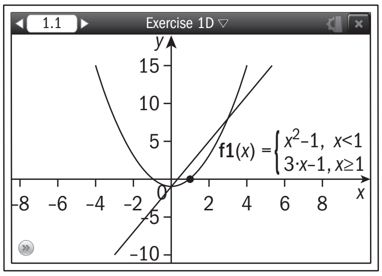
theorem $\lim\limits_{n\to\infty}\dfrac{\ln(n+k)}{n}=0$

b $\lim\limits_{n\to\infty}\sqrt[n]{n\cdot\left(\dfrac{5}{n}\right)^n}=\lim\limits_{n\to\infty}\sqrt[n]{n}\cdot\lim\limits_{n\to\infty}\dfrac{5}{n}=1\times 0=0$

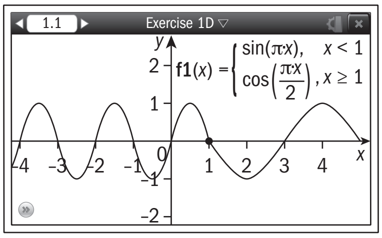
Exercise 1D

1 **a** $\lim\limits_{x\to1^-}(x^2-1)=1-1=0\ne\lim\limits_{x\to1^+}(3x-1)=3-1=2$

b



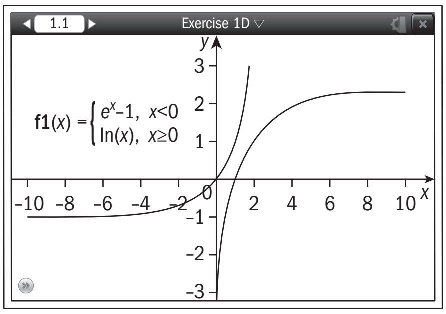
2 **a**



$\lim\limits_{x\to1^-}\sin(\pi x)=\lim\limits_{x\to1^+}\cos\left(\dfrac{\pi x}{2}\right)=0$

$\therefore\lim\limits_{x\to1}f(x)=0$

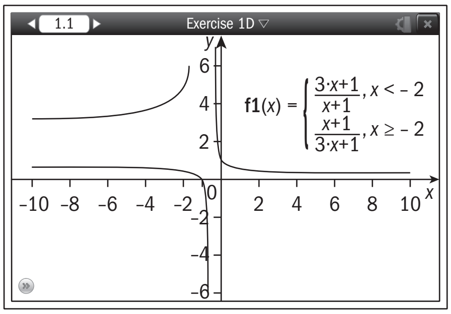
b



$$\lim_{x\to 0^-}(e^x-1)=1-1=0\neq\lim_{x\to 1^+}\ln(x)=-\infty$$

$\therefore \lim_{x\to 0} g(x)$ does not exist

c



$$\lim_{x\to -2^-}\frac{3x+1}{x+1}=\frac{-5}{-1}=5\neq\lim_{x\to -2^+}\frac{x+1}{3x+1}=\frac{-1}{-5}=\frac{1}{5}$$

$\therefore \lim_{x\to -2} h(x)$ does not exist.

3 a $$\lim_{x\to+\infty}\frac{5x-1}{2x^2+3x-1}=\lim_{x\to+\infty}\frac{\frac{5}{x}-\frac{1}{x^2}}{2+\frac{2}{x}-\frac{1}{x^2}}=\frac{0-0}{2+0+0}=0$$

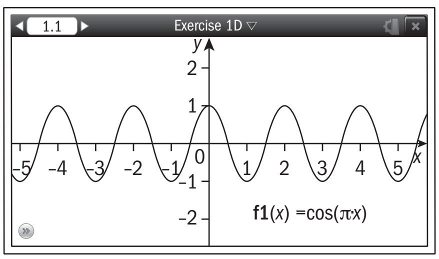
b $$\lim_{x\to+\infty}\frac{5x^2-x+1}{x^2-3x+1}=\lim_{x\to+\infty}\frac{5-\frac{1}{x}+\frac{1}{x^2}}{1-\frac{3}{x}+\frac{1}{x^2}}=\frac{5-0+0}{1-0+0}=5$$

c $$\lim_{x\to+\infty}(\sqrt{2x+1}-\sqrt{x+1})$$

$$=\lim_{x\to+\infty}\frac{(\sqrt{2x+1}-\sqrt{x+1})(\sqrt{2x+1}+\sqrt{x+1})}{\sqrt{2x+1}+\sqrt{x+1}}$$

$$=\lim_{x\to+\infty}\frac{x}{\sqrt{2x+1}+\sqrt{x+1}}=\lim_{x\to+\infty}\frac{1}{\sqrt{\frac{2}{x}+\frac{1}{x^2}}+\sqrt{\frac{1}{x}+\frac{1}{x^2}}}=0$$

4 a



b Let $a_n=2n$ and $b_n=2n+1$.

$\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=+\infty$ and

$\lim_{n\to\infty}f(a_n)=1\neq\lim_{n\to\infty}f(b_n)=-1$. Therefore

$\lim_{x\to+\infty}f(x)$ cannot exist.

Review exercise

1 a $$\lim_{n\to\infty}\frac{n^3}{2n^3-1}=\lim_{n\to\infty}\frac{1}{2-\frac{1}{n^3}}=\frac{1}{2-0}=\frac{1}{2}$$

b $\lim_{n\to\infty}\frac{n^3}{2^n}=0$ **c** $\lim_{n\to\infty}\left(\frac{2}{3}\right)^n=0$

d $$\lim_{n\to\infty}(\sqrt{n+3}-\sqrt{n})=\lim_{n\to\infty}\frac{(\sqrt{n+3}-\sqrt{n})(\sqrt{n+3}+\sqrt{n})}{\sqrt{n+5}+\sqrt{n}}$$

$$=\lim_{n\to\infty}\frac{3}{\sqrt{n+5}+\sqrt{n}}=\frac{3}{+\infty}=0$$

e $$\lim_{n\to\infty}\frac{1+2+3+\ldots+2n}{n^2}=\lim_{n\to\infty}\frac{n+2n^2}{n^2}$$

$$=\lim_{n\to\infty}\left(\frac{1}{n}+2\right)=0+2=2$$

f $$\lim_{n\to\infty}\frac{\sum_{k=0}^{n}2\cdot3^k}{5^n}=\lim_{n\to\infty}\frac{2\cdot\frac{3^{n+1}-1}{3-1}}{5^n}=\lim_{n\to\infty}\frac{3^{n+1}-1}{5^n}$$

$$=\lim_{n\to\infty}\left(3\cdot\left(\frac{3}{5}\right)^n-\frac{1}{5^n}\right)=3\times0-0=0$$

2 e.g. $u_n=-\frac{1}{n}$

3 $\frac{3}{n^2+n}\cdot(n+1)\le\sum_{k=0}^{n}\frac{3}{n^2+k}\le\frac{3}{n^2}\cdot(n+1)$ for all $n\in\mathbb{Z}^+$ as $\lim_{n\to\infty}\frac{3}{n^2+n}\cdot(n+1)=\lim_{n\to\infty}\frac{3}{n^2}\cdot(n+1)=0$ by the squeeze theorem $\lim_{n\to\infty}\sum_{k=0}^{n}\frac{3}{n^2+k}=0$

4 $$\frac{n(n-1)}{4n^2+3}$$

$$=\sum_{k=1}^{n}\frac{n-1}{4n^2+3}\le\sum_{k=1}^{n}\frac{n+\cos(k\pi)}{4n^2+3}\le\sum_{k=1}^{n}\frac{n+1}{4n^2+3}$$

$=\frac{n(n+1)}{4n^2+3}$ for all $n\in\mathbb{Z}^+$;

as $\lim_{n\to\infty}\frac{n(n-1)}{4n^2+3}=\lim_{n\to\infty}\frac{1-\frac{1}{n}}{4+\frac{3}{n^2}}=\frac{1}{4}$ and

$\lim_{n\to\infty}\frac{n(n+1)}{4n^2+3}=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{4+\frac{3}{n^2}}=\frac{1}{4}$ by the squeeze theorem $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n+\cos(k\pi)}{4n^2+3}=\frac{1}{4}$ which is a real number. Therefore the sequence converges.

5 **a** $\frac{1}{3n(3n-1)}, n \in \mathbb{Z}^+$

b 1 **c** 0

6 $\frac{n!}{(n+2)!} \le \frac{n!+\sqrt{n}}{(n+2)!} \le \frac{2n!}{(n+2)!}$ for all $n \in \mathbb{Z}^+$;

as $\lim_{n\to\infty}\frac{n!}{(n+2)!} = \lim_{n\to\infty}\frac{n!}{(n+2)(n+1)n!}$

$= \lim_{n\to\infty}\frac{1}{(n+2)(n+1)} = 0$

and $= \lim_{n\to\infty}\frac{2n!}{(n+2)!}$

$= \lim_{n\to\infty}\frac{2n!}{(n+2)(n+1)n!}$

$= \lim_{n\to\infty}\frac{2}{(n+2)(n+1)} = 0$

by the squeeze theorem $\lim_{n\to\infty}\frac{n!+\sqrt{n}}{(n+2)!} = 0$

7 **a** $a_1 = 3 > 0$ and each term is obtained from the previous one by adding 1 unit and then dividing the sum by 2. Therefore all the terms remain positive.

Note: you can also show it using mathematical induction.

b Let $P(n): a_{n+1} - a_n < 0$.

Step 1: $P(1): a_2 - a_1 < 0$ which is true as

$$a_2 = \frac{3+1}{2} = 2 \text{ and } a_2 - a_1 = 2 - 3 = -1 < 0$$

Step 2: Assume $P(k): a_{k+1} - a_k < 0$ true and consider $P(k+1): a_{k+2} - a_{k+1} < 0$.

As $a_{k+2} - a_{k+1} = \frac{a_{k+1}+1}{2} - \frac{a_k+1}{2} = \frac{1}{2}(a_{k+1} - a_k) = < 0$.

Therefore, $P(k)$true $\Rightarrow P(k+1)$true.

By the PMI, as $P(1)$true and $P(k)$true $\Rightarrow P(k+1)$true, we can conclude that $P(n)$true for all $n \in \mathbb{Z}^+$.

c $a_n = \left(\frac{1}{2}\right)^{n-2} + 1$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\left(\frac{1}{2}\right)^{n-2} + 1\right) = 1$

d $\lim_{n\to\infty}(a_n)^n = \lim_{n\to\infty}\left(\left(\frac{1}{2}\right)^{n-2} + 1\right)^n$

$$= \lim_{n\to\infty}\Bigg(\underbrace{\left(\frac{1}{2}\right)^{n(n-2)} + n\cdot\left(\frac{1}{2}\right)^{(n-1)(n-2)} + \ldots + 1}_{u_n}\Bigg)$$

(using binomial expansion).

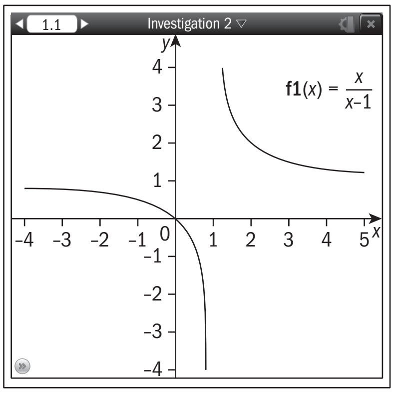
As $1 \le u_n \le \left(\frac{1}{2}\right)^{n(n-2)}\underbrace{\sum_{k=0}^{n-1}\binom{n}{k}}_{2^n-1} + 1 = \frac{2^n-1}{2^n}\left(\frac{1}{2}\right)^{n-2} + 1$

and $\frac{2^n-1}{2^n}\left(\frac{1}{2}\right)^{n-2} + 1$, by the squeeze theorem $\lim_{n\to\infty}(a_n)^n = 1$.

2 Smoothness in mathematics

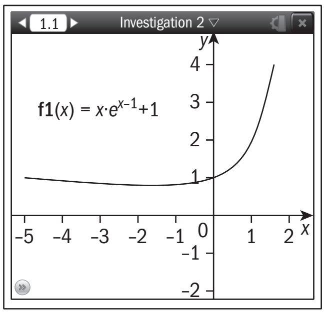
'Before you start'

1 a



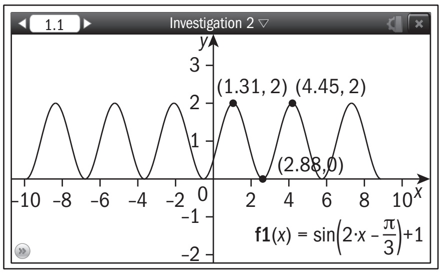
Asymptotes: $x = 1$, $y = 1$

b



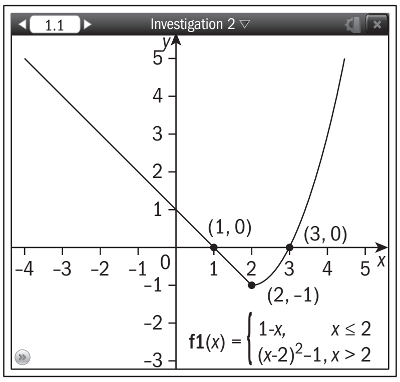
Asymptote: $y = 1$

c



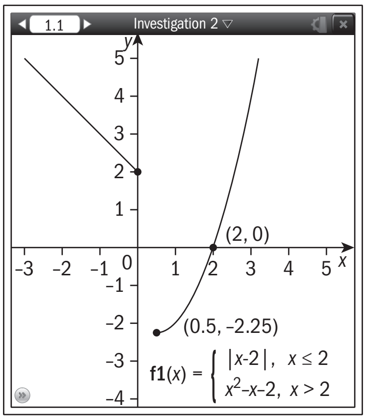
no asymptotes

2 a



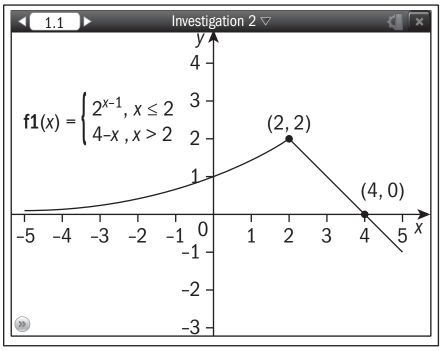
no asymptotes

b



no asymptotes

c



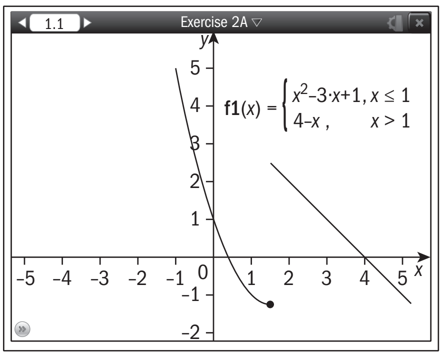
Asymptote: $y = 0$

3 a $\lim_{x \to 1}\left((4x-1)^2-5\right) = (4-1)^2 - 5 = 9 - 5 = 4$

b $\lim_{x \to 0}\left(\frac{\tan(2x)}{2x} - 3\right)^3 = (1-3)^3 = (-2)^3 = -8$

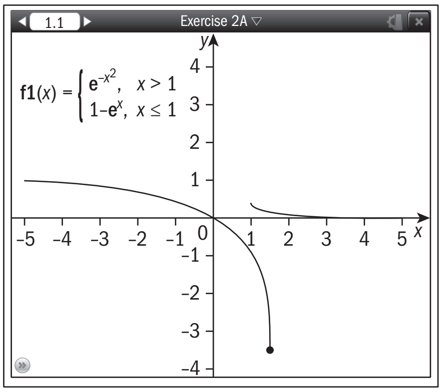
Exercise 2A

1 a



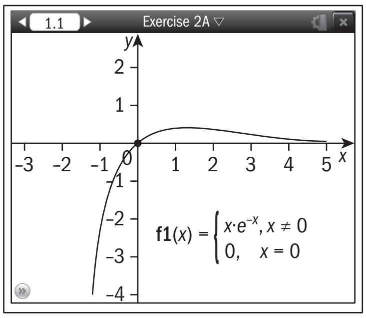
discontinuous at $x = 1$

b



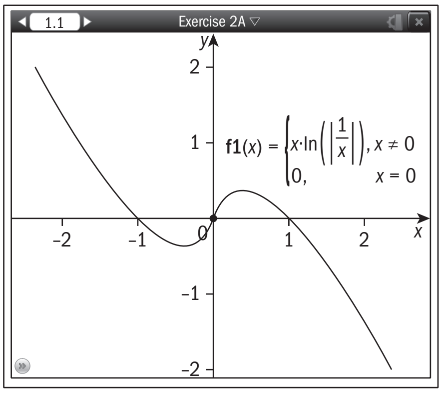
discontinuous at $x = 1$

c



continuous in $\mathbb{R}$

d



continuous in $\mathbb{R}$

2 Consider the sequences $u_n = \dfrac{1}{2\pi n}$ and $v_n = \dfrac{1}{\frac{\pi}{2} + \pi n}$

$\lim\limits_{x\to\infty} g(u_n) = 1 \neq \lim\limits_{x\to\infty} g(v_n) = 0.$

$\lim\limits_{x\to\infty} g(x)$ does not exist

3 Apply Squeeze theorem:

$\lim\limits_{x\to 0}\left(x\sin\left(\dfrac{1}{x}\right)\right) = 0 = f(0)$ as

$-1 \le \sin\left(\dfrac{1}{x}\right) \le 1$ for all $x \in \mathbb{R}$.

4 a $\lim\limits_{x\to a}(f+g)(x) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$

$= f(a) + g(a) = (f+g)(a).$

b $\lim\limits_{x\to a}(f-g)(x) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$

$= f(a) - g(a) = (f-g)(a).$

c $\lim\limits_{x\to a}(f\cdot g)(x) = \lim\limits_{x\to a} f(x)\cdot\lim\limits_{x\to a} g(x)$

$= f(a)\cdot g(a) = (f\cdot g)(a).$

d $\lim\limits_{x\to a}\left(\dfrac{f}{g}\right)(x) = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{f(a)}{g(a)} = \left(\dfrac{f}{g}\right)(a)$

for $g(a) \neq 0.$

Investigation –Composition of continuous functions

a $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}(x+1) = 1 \neq f(0) = 0$

and $\lim\limits_{x\to 1} g(x) = \lim\limits_{x\to 1}(x) = 1 \neq g(1) = 0$

but $\lim\limits_{x\to 0} f(g(x)) = \lim\limits_{x\to 0} f(0) = 0 = f(g(0))$

and $\lim\limits_{x\to 1} g(f(x)) = \lim\limits_{x\to 1} g(2) = 2 = g(f(1))$

b $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1}(x+1) = 2 = f(1)$

and $\lim\limits_{x\to 1} h(x) = \lim\limits_{x\to 1}(x-1) = 0 = h(1)$

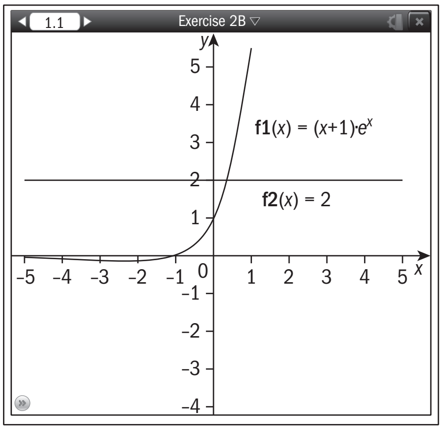
but $\lim\limits_{x\to 1^+} h(f(x)) = \lim\limits_{x\to 1^+} h(x+1) = 3 \neq \lim\limits_{x\to 1^-} h(f(x))$

$= \lim\limits_{x\to 1^-} h(x+1) = 1$

c g is continuous at $x = a$ and f is defined and continuous at $x = g(a)$.

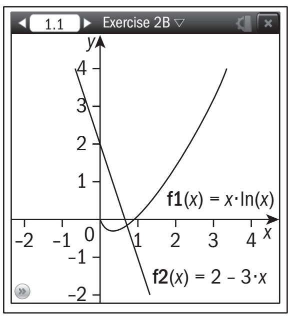
Exercise 2B

1 a



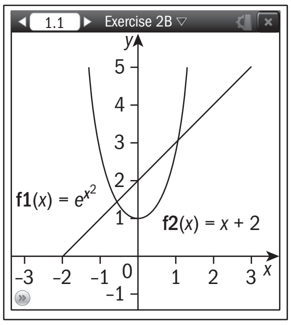
Apply theorem 5 (Bolzano's) to a closed interval that contains the point of intersection of the graphs (e.g. [0, 1]).

b



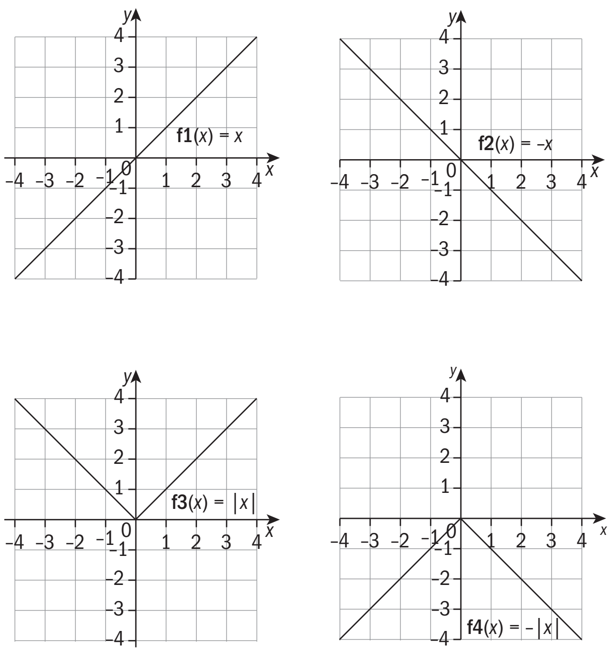
Apply theorem 5 (Bolzano's) to a closed interval that contains the point of intersection of the graphs (e.g. [0, 1]).

c



Apply theorem 5 (Bolzano's) to closed intervals that contains the point of intersection of the graphs (e.g. $[-1, 0]$ and $[0, 2]$).

2



Reasons: $(f(x))^2 = x^2 \Rightarrow f(x) = \pm x$ and, as f is continuous, the function can just change from $f(x) = x$ to $f(x) = -x$ at $x = 0$, so we also have $f(x) = |x|$ and $f(x) = -|x|$.

3 Suppose that $f(x_1) \neq f(x_2)$, let's say $f(x_1) < f(x_2)$ for $x_1, x_2 \in [a, b]$. Let $y \in \left]f(x_1), f(x_2)\right[\cap (\mathbb{R} \setminus \mathbb{Q})$. By Bolzano's theorem, there must exist at least a value $x \in \left]x_1, x_2\right[$ such that $f(x) = y$. But $y \notin \mathbb{Q}$ which contradicts the condition that $f(x) \in \mathbb{Q}$ for all values $x \in [a, b]$.
c can be any rational number.

4 Let $p(x)$ be a polynomial with odd degree. Then $\lim_{x\to-\infty} p(x) = -\lim_{x\to+\infty} p(x)$ and by Bolzano's theorem the polynomial must have at least a zero in $]-\infty, +\infty[$.

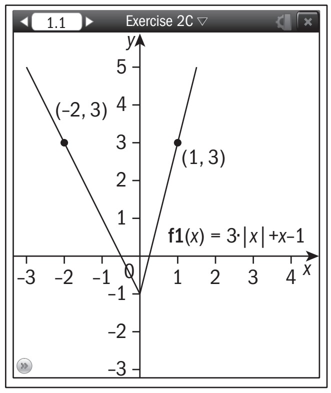
Exercise 2C

1 **a** $f(-2) = 3|-2| - 2 - 1 = 3$ and $f(1) = 3|1| + 1 - 1 = 3$

b $f'(x) = \begin{cases} 4, & x > 0 \\ -2, & x < 0 \end{cases}$

This function clearly has no zeros in any interval.

c



This function is not differentiable at $x = 0$, so we cannot use Rolle's Theorem.

2 **a** $f(-1) = 13$ and $f(1) = -3$. By Bolzano's theorem, as the function changes sign on the interval $[-1, 1]$, it has at least a zero on $]-1, 1[$.

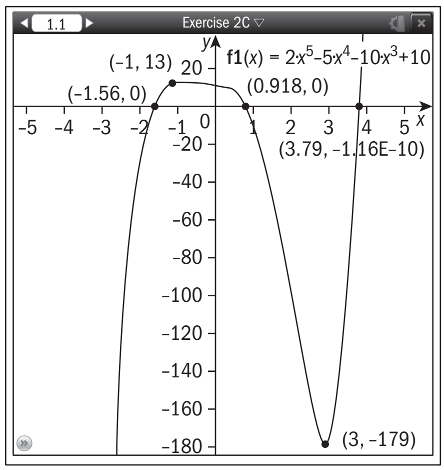
b $f'(x) = 10x^4 - 20x^3 - 30x^2$

$f'(x) = 0 \Rightarrow 10x^2(x^2 - 2x - 3) = 0$

$\Rightarrow x = 0, x = -1, x = 3$

c If f had another zero on $]-1, 1[$, then its derivative would have also a zero on this interval which contradict the result in part **b**.

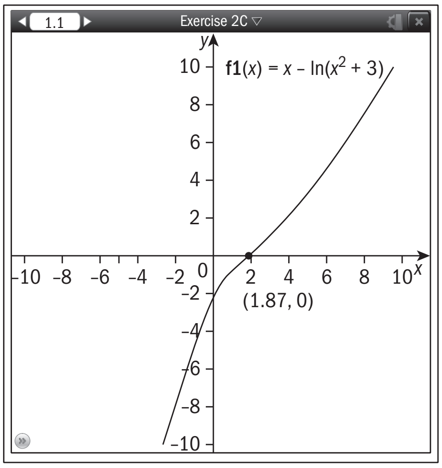
d



3 **a** $f'(x) = 1 - \dfrac{2x}{x^2+3} = \dfrac{x^2 - 2x + 3}{x^2+3} = \dfrac{(x-1)^2 + 2}{x^2+3} > 0$ for all $x \in \mathbb{R}$.

b If the equation $f(x) = 0$ had more than a real solution, let's say, $a < b$ were both real solutions of $f(x) = 0$, then by Rolle's theorem $f'(x) = 0$ would have at least a solution on the interval $]a, b[$ which contradicts the result proved in part **a**.

c



No maximun nor minimum points

4 Let $f(x) = x^3 + 3x - 2$. This function is continuous and differentiable on any interval of real numbers. As $f(0) = -2 < 0$ and $f(1) = 2 > 0$ by Bolzano's theorem the function must have at least a zero on $]0, 1[$. As $f'(x) = 3x^2 + 3 > 0$ for all $x \in \mathbb{R}$ and therefore f' has no real zeros, the function f cannot have more than one real zero.

5 a Let $f(x) = \sin(x)$ for $0 < x < \frac{\pi}{2}$. Then $f'(x) = \cos(x)$, for $0 < x < \frac{\pi}{2}$.

f is continuous and differentiable for $0 < x < \frac{\pi}{2}$.

Therefore by the MVT, there is at least one value of $c \in]0, x[$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \Rightarrow f'(c) = \frac{\sin(x)}{x}.$$

b As $0 < c < x < \frac{\pi}{2}$ and $f'(c) = \frac{\sin(x)}{x}$

$\Rightarrow \cos(c) = \frac{\sin(x)}{x}$ it follows that

$\cos(x) < \cos(c) < \cos(0) \Rightarrow \cos(x) < \frac{\sin(x)}{x} < 1$ for $0 < x < \frac{\pi}{2}$.

6 a Let $f(x) = e^x$ for $x > 0$. Then $f'(x) = e^x$, for $x > 0$.

f is continuous and differentiable for $x > 0$.

Therefore by the MVT, there is at least one value of $c \in]0, +\infty[$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \Rightarrow f'(c) = \frac{e^x - 1}{x - 0}.$$

As $0 < c < x$ and $f'(c) = \frac{e^x - 1}{x - 0} \Rightarrow \frac{e^x - 1}{x} = e^c > 1$ it follows that

$\frac{e^x - 1}{x} > 1 \Rightarrow e^x - 1 > x \Rightarrow e^x > x + 1$ for $x > 0$.

b Let $f(x) = \ln(x)$ for $x > 0$. Then $f'(x) = \frac{1}{x}$, for $x > 0$.

f is continuous and differentiable for $x > 0$.

Therefore by the MVT, there is at least one value of $c \in]x, x + 1[$ such that

$$f'(c) = \frac{f(x+1) - f(x)}{(x+1) - x} \Rightarrow f'(c) = \frac{\ln(x+1) - \ln(x)}{1} \Rightarrow f'(c) = \ln\left(\frac{x+1}{x}\right).$$

As $x + 1 > c > x > 0 \Rightarrow \frac{1}{x+1} < \frac{1}{c} < \frac{1}{x}$ and $f'(c) = \frac{1}{c} = \ln\left(\frac{x+1}{x}\right)$ it follows that $\ln\left(\frac{x+1}{x}\right) < \frac{1}{x}$ for $x > 0$.

c Let $f(x) = \tan(x)$ for $0 < x < \frac{\pi}{2}$. Then $f'(x) = \frac{1}{\cos^2(x)}$, for $0 < x < \frac{\pi}{2}$.

f is continuous and differentiable for $0 < x < \frac{\pi}{2}$.

Therefore by the MVT, there is at least one value of $c \in]0, x[$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \Rightarrow f'(c) = \frac{\tan(x) - \tan(0)}{x} \Rightarrow f'(c) = \frac{\tan(x)}{x}.$$

As $0 < c < x < \frac{\pi}{2} \Rightarrow \cos^2(c) < 1 \Rightarrow \frac{1}{\cos^2(c)} > 1$ and $f'(c) = \frac{1}{\cos^2(c)}$ it follows that

$\frac{\tan(x)}{x} > 1 \Rightarrow \tan(x) > x$ for $0 < x < \frac{\pi}{2}$.

d Let $f(x) = \arcsin(x)$ for $0 < x < 1$. Then $f'(x) = \frac{1}{\sqrt{1 - x^2}}$, for $0 < x < 1$.

f is continuous and differentiable for $0 < x < 1$.

Therefore by the MVT, there is at least one value of $c \in]0, x[$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0} \Rightarrow f'(c) = \frac{\arcsin(x) - \arcsin(0)}{x} \Rightarrow f'(c) = \frac{\arcsin(x)}{x}.$$

As $0 < c < x < 1$ and $f'(c) = \frac{1}{\sqrt{1 - c^2}} > 1$ it follows that

$\frac{\arcsin(x)}{x} > 1 \Rightarrow \arcsin(x) > x$ for $0 < x < 1$.

7 We need to show that $f'(x) = g'(x)$ and it follows from corollary 2 of Mean Value Theorem that $g(x) = f(x) + c$ for some real constant c.

$$g'(x) = \frac{\frac{2}{(1-x)^2}}{\left(\frac{1+x}{1-x}\right)^2 + 1} = \frac{\frac{2}{(1-x)^2}}{\frac{(1+x)^2+(1-x)^2}{(1-x)^2}}$$

$$= \frac{2}{2+2x^2} = \frac{1}{1+x^2} = f'(x)$$

8 We need to show that $f'(x) > 0$ for $x \in]0, 2[$ and it follows from corollary 3 of Mean Value Theorem that f is increasing on $]0, 2[$.

$f'(x) = 2x\,e^{-x} - x^2e^{-x} = (2x - x^2)e^{-x} > 0$ when $2x - x^2 > 0$, i.e., for $x \in]0, 2[$.

9 Consider $h(x) = f(x) - g(x)$.

$f'(x) = g'(x) \Rightarrow h'(x) = 0$.

By corollary 1 of Mean Value Theorem to $h(x) = f(x) - g(x)$ and it follows that $h(x) = c \Rightarrow f(x) - g(x) = c \Rightarrow f(x) = g(x) + c$.

QED

10 Consider any two points $a < b$ in I. Then

$$f'(x) = \frac{f(b) - f(a)}{b - a} < 0 \Rightarrow f(b) - f(a) < 0$$

$$\Rightarrow f(b) < f(a)$$

Therefore, for any two values $a < b$ which means that f is decreasing on I.

QED

Investigation

a e.g.

x1=	2			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	3	1.767766953	1.732050808	2.062072616
2	4	2.121320344	2	6.066017178
…	…	…	…	…
9	11	4.596194078	3.31662479	38.58046563
10	12	4.949747468	3.464101615	42.88690166

x1=	2			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	2.1	1.449568901	1.449137675	0.029757477
2	2.2	1.48492424	1.483239697	0.113571871
…	…	…	…	…
9	2.9	1.732411614	1.702938637	1.730712821
10	3	1.767766953	1.732050808	2.062072616

x1=	2			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	2.01	1.417749096	1.417744688	0.000310945
2	2.02	1.42128463	1.42126704	0.001237616
…	…	…	…	…
9	2.09	1.446033368	1.445683229	0.024219555
10	2.1	1.449568901	1.449137675	0.029757477

x1=	5			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	5.01	2.238304045	2.238302929	4.99002E–05
2	5.02	2.240540113	2.24053565	0.000199203
…	…	…	…	…
9	5.09	2.256192589	2.256102835	0.00397831
10	5.1	2.258428657	2.258317958	0.004901841

For this function the error decreases as h decreases and as x_1 increases.

b

x1=	2			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	3	14.7781122	20.08553692	26.42411177
2	4	22.1671683	54.59815003	59.39941503
…	…	…	…	…
9	11	73.89056099	59874.14172	99.8765902
10	12	81.27961709	162754.7914	99.95006008

x1=	2			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	2.01	7.46294666	7.463317347	0.004966791
2	2.02	7.536837221	7.538324934	0.019735323
…	…	…	…	…
9	2.09	8.054071148	8.084915164	0.381500805
10	2.1	8.127961709	8.166169913	0.467884016

x1=	5			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	5.01	149.8972907	149.9047361	0.004966791
2	5.02	151.3814223	151.4113038	0.019735323
…	…	…	…	…
9	5.09	161.7703434	162.3898621	0.381500805
10	5.1	163.254475	164.0219073	0.467884016

x1=	10			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	10.01	22246.73045	22247.83546	0.004966791
2	10.02	22466.99511	22471.42992	0.019735323
…	…	…	…	…
9	10.09	24008.84772	24100.79243	0.381500805
10	10.1	24229.11237	24343.00942	0.467884016

For this function the error decreases as h decreases but does not seem to depend on x_1.

c

x1=	2			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	3	1.193147181	1.098612289	8.604936688
2	4	1.693147181	1.386294361	22.13475204
…	…	…	…	…
9	11	5.193147181	2.397895273	116.5710588
10	12	5.693147181	2.48490665	129.1090968

x1=	2			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	2.1	0.743147181	0.741937345	0.16306442
2	2.2	0.793147181	0.78845736	0.594809616
…	…	…	…	…
9	2.9	1.143147181	1.064710737	7.366925198
10	3	1.193147181	1.098612289	8.604936688

x1=	2			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	2.01	0.698147181	0.698134722	0.001784539
2	2.02	0.703147181	0.703097511	0.007064333
…	…	…	…	…
9	2.09	0.738147181	0.737164066	0.13336442
10	2.1	0.743147181	0.741937345	0.16306442

x1=	5			
n	x2	f(x2) appro	f(x2)	Relative Error (%)
1	5.01	1.611437912	1.611435915	0.000123948
2	5.02	1.613437912	1.613429934	0.00049452
…	…	…	…	…
9	5.09	1.627437912	1.627277831	0.009837403
10	5.1	1.629437912	1.62924054	0.012114399

For this function the error decreases as h decreases and decreases as x_1 increases.

Exercise 2D

1 a $\lim\limits_{x\to0}\frac{x^2-3x}{\sin x}=\lim\limits_{x\to0}\frac{2x-3}{\cos x}=-\frac{3}{1}=-3$

b $\lim\limits_{x\to0}\frac{e^{2x}-e^{-3x}}{x}=\lim\limits_{x\to0}\frac{2e^{2x}+3e^{-3x}}{1}=5$

c $\lim\limits_{x\to0}\frac{\cos(x)+2x-1}{3x}=\lim\limits_{x\to0}\frac{-\sin(x)+2}{3}=\frac{2}{3}$

d $\lim\limits_{x\to0}\frac{e^{2x}-e^{-x}-2}{1-\cos(2x)}=\lim\limits_{x\to0}\frac{2e^{2x}+e^{-x}}{2\sin(2x)}=\frac{3}{0}=\infty$

e $\lim\limits_{x\to\frac{\pi}{2}^+}\frac{\ln(\cot(x))}{\ln(\cos(x))}=\lim\limits_{x\to\frac{\pi}{2}^+}\frac{\frac{-\csc^2(x)}{\cot(x)}}{\frac{-\sin(x)}{\cos(x)}}=\lim\limits_{x\to\frac{\pi}{2}^+}\frac{\frac{-\csc^2(x)}{\cot(x)}}{\frac{-\sin(x)}{\cos(x)}}=1$

f $\lim\limits_{x\to0}\frac{x(1-e^x)}{\ln(\cos x)}=\lim\limits_{x\to0}\frac{1-e^x+xe^x}{\frac{-\sin x}{\cos x}}=\lim\limits_{x\to0}\frac{\cos x(1-e^x+xe^x)}{-\sin x}$

$=\lim\limits_{x\to0}\frac{-\sin x(1-e^x+xe^x)+\cos x(-e^x+e^x+xe^x)}{-\cos x}=0$

g $\lim\limits_{x\to0}\frac{x-\arcsin x}{\sin^2 x}=\lim\limits_{x\to0}\frac{1-\frac{1}{\sqrt{1-x^2}}}{\underbrace{2\sin x\cos x}_{\sin(2x)}}=\lim\limits_{x\to0}\frac{\frac{2x}{2\sqrt{1-x^2}}}{2\cos(2x)}=0$

h $\lim\limits_{x\to0^+}\frac{e^{2x}-\cos(3x)}{x\sin(2x)}=\lim\limits_{x\to0}\frac{2e^{2x}+3\sin(3x)}{\sin(2x)+x\cos(2x)}=\frac{2}{0^+}=+\infty$

i $\lim\limits_{x\to\pi}\frac{1+\cos(x)}{x^2\sin(x)}=\lim\limits_{x\to\pi}\frac{-\sin(x)}{2x\sin(x)+x^2\cos(x)}=0$

2 a $\lim\limits_{x\to1}\frac{x^2+3x-2}{x^2-x+2}=\frac{2}{2}=1\neq6$

b $\lim\limits_{x\to1}\frac{1-\cos(x)}{x^2+x}=\frac{1-\cos(1)}{2}\neq\frac{\cos(1)}{2}$

In both parts *a* and *b* there is no indeterminate form and therefore L'Hôpital Rule cannot be applied, ie, this is not a valid method to find these limits.

3 a $\lim\limits_{x\to+\infty}\frac{e^{5x}+x}{4x+1}=\lim\limits_{x\to+\infty}\frac{5e^{5x}+1}{4}=+\infty$

b $\lim\limits_{x\to+\infty}\frac{x^2}{x(e^x+1)}=\lim\limits_{x\to+\infty}\frac{x}{e^x+1}=\lim\limits_{x\to+\infty}\frac{1}{e^x}=\frac{1}{+\infty}=0$

c $\lim\limits_{x\to+\infty}\frac{\ln(e^x+1)}{x+3}=\lim\limits_{x\to+\infty}\frac{\frac{e^x}{e^x+1}}{1}=\lim\limits_{x\to+\infty}\frac{e^x}{e^x}=1$

4 a $f(x)=\frac{1}{x}\Rightarrow f'(x)=-\frac{1}{x^2}$

$\therefore f$ is continuous and differentiable in any interval

of its domain $\mathbb{R}\setminus\{0\}$

$g(x)=(x-1)^3\Rightarrow g'(x)=3(x-1)^2$

$\therefore g$ is continuous and differentiable in any interval of its domain $\mathbb{R}$

b Applying Cauchy's theorem (corollary of MTV) to *f* and *g* on the interval [1,3]

$-\frac{2}{3}\times3(x-1)^2=-\frac{8}{x^2}\Rightarrow x^2(x-1)^2=4$

$\Rightarrow x(x+1)=\pm2\Rightarrow x^2-x+2=0$ or $x^2-x-2=0$

$\Rightarrow \underbrace{x=\frac{1\pm\sqrt{1-8}}{2}}_{\text{no real solutions}}$ or $x=\frac{1\pm\sqrt{1+8}}{2}\Rightarrow x=2$ or $x=-1$

Therefore there is exactly one solution on the interval]1, 3[

5 a $f(x)=x^3-3x\Rightarrow f'(x)=3x^2-3$

$\therefore f$ is continuous and differentiable in any interval of its domain $\mathbb{R}$

$g(x)=(x-1)^2\Rightarrow g'(x)=2(x-1)=2x-2$

$\therefore g$ is continuous and differentiable in any interval of its domain $\mathbb{R}$

b Applying Cauchy's theorem (corollary of MTV) to f and g on the interval [–4, 2] we would conclude that the equation had at least on solution in the interior of the interval.

c $\frac{2(x-1)}{3x^2-3} = \frac{-24}{64} \Rightarrow 16(x-1) = -9(x^2-3)$

$\Rightarrow 9x^2 - 16x - 43 = 0$ which has no real solution.

d Cauchy's theorem cannot be applied in part b because f' has zeros on the interval given.

Exercise 2E

1 a Indeterminate form: $\infty \times 0$;

$$\lim_{x\to 0^+} \frac{3}{x} e^{-\frac{2}{x}} = \lim_{x\to 0^+} \frac{\frac{3}{x}}{e^{\frac{2}{x}}} = \lim_{x\to 0^+} \frac{-\frac{3}{x^2}e}{-\frac{2}{x^2}e^{\frac{2}{x}}} = 0$$

b Indeterminate form: $\infty \times 0$;

$$\lim_{x\to 1^+} \arctan(x-1)e^{\frac{3}{x-1}} = \lim_{x\to 1^+} \frac{\arctan(x-1)}{e^{-\frac{3}{x-1}}}$$

$$= \lim_{x\to 1^+} \frac{\frac{1}{(x-1)^2+1}}{\frac{3}{(x-1)^2} e^{-\frac{3}{x-1}}} = \lim_{x\to 1^+} \frac{1}{(x-1)^2+1} \bullet \lim_{x\to 1^+} \frac{e^{\frac{3}{x-1}}}{\frac{3}{(x-1)^2}}$$

$$\lim_{x\to 1^+} \frac{-\frac{3}{(x-1)^2} e^{\frac{3}{x-1}}}{-\frac{6}{(x-1)^3}} = \lim_{x\to 1^+} \frac{e^{\frac{3}{x-1}}}{\frac{2}{(x-1)}} = \lim_{x\to 1^+} \frac{-\frac{9}{(x-1)^2} e^{\frac{3}{x-1}}}{-\frac{6}{(x-1)^2}} = \infty$$

c Indeterminate form: $\infty \times 0$

$$\lim_{x\to 0^+} (\sin(x))(\ln(x) - \cot(x)) = \lim_{x\to 0^+} \frac{\ln(x) - \cot(x)}{\frac{1}{\sin(x)}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x} + \frac{1}{\sin^2(x)}}{-\frac{\cos(x)}{\sin^2(x)}} = \lim_{x\to 0^+} \frac{\frac{\sin^2(x) - x}{x\sin^2(x)}}{-\frac{\cos(x)}{\sin^2(x)}}$$

$$= \lim_{x\to 0^+} \frac{x - \sin^2(x)}{x\cos(x)} = \lim_{x\to 0^+} \frac{1 - 2\sin(x)\cos(x)}{\cos(x) - x\sin(x)} = 1$$

d Indeterminate form: 1^∞

$$\lim_{x\to 0^+} (x + e^x)^{\frac{2}{x}} = e^{\lim_{x\to 0^+} \ln(x+e^x)^{\frac{2}{x}}} \text{ and}$$

$$\lim_{x\to 0^+} \ln(x+e^x)^{\frac{2}{x}} = 2\lim_{x\to 0^+} \frac{\ln(x+e^x)}{x}$$

$$= 2\lim_{x\to 0^+} \frac{\frac{1+e^x}{x+e^x}}{1} = 4$$

$$\therefore \lim_{x\to 0^+} (x+e^x)^{\frac{2}{x}} = e^4$$

e Indeterminate form: 0^0

$$\lim_{x\to 0^+} (\sin(2x))^{3x} = e^{\lim_{x\to 0^+} \ln(\sin(2x))^{3x}} \text{ and}$$

$$\lim_{x\to 0^+} \ln(\sin(2x))^{3x} = \lim_{x\to 0^+} \frac{\ln(\sin(2x))}{\frac{1}{3x}} = \lim_{x\to 0^+} \frac{\frac{2\cos(2x)}{\sin(2x)}}{-\frac{1}{3x^2}}$$

$$= \lim_{x\to 0^+} \left(-\frac{6x^2\cos(2x)}{\sin(2x)}\right)$$

$$= \lim_{x\to 0^+} \left(-\frac{12x\cos(2x) + 12x^2\sin(2x)}{2\cos(2x)}\right) = 0$$

$$\therefore \lim_{x\to 0^+} (\sin(2x))^{3x} = e^0 = 1$$

f Indeterminate form: 1^∞

$$\lim_{x\to 0^+} (1+x)^{\frac{5}{x}} = e^{\lim_{x\to 0^+} \ln(1+x)^{\frac{5}{x}}} \text{ and } \lim_{x\to 0^+} \ln(1+x)^{\frac{5}{x}}$$

$$= 5\lim_{x\to 0^+} \frac{\ln(1+x)}{x} = 5\lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = 5$$

$$\therefore \lim_{x\to 0^+} (1+x)^{\frac{5}{x}} = e^5$$

g Indeterminate form: $\infty \times 0$

$$\lim_{x\to 0} (2x\cot(5x)) = \lim_{x\to 0} \frac{2x}{\tan(5x)} = \lim_{x\to 0^+} \frac{2}{\frac{5}{\cos^2(5x)}} = \frac{2}{5}$$

h Indeterminate form: 1^∞

$$\lim_{x\to 0} \ln(e-x)^{\cot(x)} = e^{\lim_{x\to 0} \cot(x)\ln(1+x)} \text{ and}$$

$$\lim_{x\to 0} \cot(x)\ln(e-x) = \lim_{x\to 0} \frac{\ln(e-x)}{\tan(x)}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{e-x}}{\frac{1}{\cos^2(x)}} = \lim_{x\to 0} \left(-\frac{\cos^2(x)}{e-x}\right) = -\frac{1}{e}$$

$$\therefore \lim_{x\to 0} \ln(e-x)^{\cot(x)} = e^{-\frac{1}{e}}$$

i Indeterminate form: 1^∞

$$\lim_{x\to 0} (e^{e^x-1})^{\frac{1}{x}} = e^{\lim_{x\to 0} \frac{\ln(e^{e^x-1})}{x}} = e^{\lim_{x\to 0} \frac{e^x-1}{x}} = e^{\lim_{x\to 0} \frac{e^x}{1}} = e$$

k Indeterminate form: $\infty - \infty$

$$\lim_{x\to 0^+} (\ln(x) - \ln(\sin(x))) = \ln\left(\lim_{x\to 0^+}\left(\frac{x}{\sin x}\right)\right)$$

$$= \ln\left(\lim_{x\to 0^+}\left(\frac{1}{\cos x}\right)\right) = \ln 1 = 0$$

l Indeterminate form: $\infty \times 0$

$$\lim_{x\to -\infty} (x\,e^x) = \lim_{x\to -\infty} \frac{x}{e^{-x}} = \lim_{x\to -\infty} \frac{1}{-e^{-x}} = 0$$

m Indeterminate form: $\infty - \infty$

$$\lim_{x\to0^+}\left(\ln(e^x+1)-\ln(e^x-1)\right)=\ln\left(\lim_{x\to0^+}\frac{e^x+1}{e^x-1}\right)$$

$$=\ln\left(\lim_{x\to0^+}\left(\frac{e^x}{e^x}\right)\right)=\ln 1=0$$

n Indeterminate form: $\infty \times 0$

$$\lim_{x\to+\infty}\left(x\sin\left(\frac{1}{x}\right)\right)=\lim_{x\to+\infty}\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$$

$$=\lim_{x\to+\infty}\frac{-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)}{-\frac{1}{x^2}}=\lim_{x\to+\infty}\cos\left(\frac{1}{x}\right)=1$$

o Indeterminate form: ∞^0

$$\lim_{x\to+\infty}(1+x)^{\frac{1}{x}}=e^{\lim_{x\to+\infty}\ln(1+x)^{\frac{1}{x}}}\text{ and}$$

$$\lim_{x\to+\infty}\ln(1+x)^{\frac{1}{x}}=\lim_{x\to+\infty}\frac{\ln(1+x)}{x}=\lim_{x\to+\infty}\frac{\frac{1}{1+x}}{1}=0$$

$$\therefore \lim_{x\to+\infty}(1+x)^{\frac{1}{x}}=e^0=1$$

2 a $a=\lim_{x\to0}\left(\frac{\sin(3x)}{x}\right)=\lim_{x\to0}\left(\frac{3\cos(3x)}{1}\right)=3$

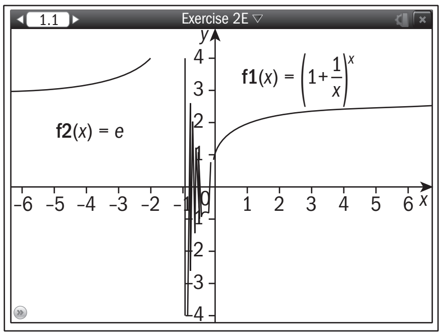
b $\lim_{x\to0^-}(b-x)=\lim_{x\to0^+}(e^x+4x)^{\frac{1}{x}}$

$$\Rightarrow b=e^{\lim_{x\to0^+}\frac{\ln(e^x+4x)}{x}}=e^{\lim_{x\to0^+}\frac{\frac{e^x+4}{e^x+4x}}{1}}=e^5$$

c $c=\lim_{x\to0}\left(\frac{1-\cos(x)}{x^2}\right)=\lim_{x\to0}\left(\frac{3\sin(x)}{2x}\right)$

$$=\lim_{x\to0}\left(\frac{-3\cos(x)}{2}\right)=-\frac{3}{2}$$

3 a



; $\lim_{x\to+\infty} f(x)=e$

b $g(h)=\frac{\ln(1+h)}{h}$

c $\lim_{h\to0^+}\frac{\ln(1+h)}{h}=\lim_{h\to0^+}\frac{\frac{1}{1+h}}{1}=1.$

Therefore $\lim_{x\to+\infty} f(x)=e^{\lim_{h\to0^+}g(h)}=e^1=e$

Exercise 2F

1 $\lim_{x\to2^+}(ax^2-x)=\lim_{x\to2^-}(x^3+bx)$

$\Rightarrow 4a-2=8+2b\Rightarrow 2a-b=5$

and $\lim_{x\to2^+}(2ax-1)=\lim_{x\to2^-}(3x^2+b)$

$\Rightarrow 4a-1=12+b\Rightarrow 4a-b=13$

Solving simultaneously we obtain $a=4$, $b=3$

2 $\lim_{x\to0^+}\frac{e^{ax}-1}{x}=\lim_{x\to0^-}(bx+3)\Rightarrow\lim_{x\to0^+}\frac{ae^{ax}}{1}=3\Rightarrow a=3$

and $\lim_{x\to0^+}\frac{axe^{ax}-(e^{ax}-1)}{x^2}=\lim_{x\to0^-}b$

$\Rightarrow\lim_{x\to0^+}\frac{ae^{ax}+a^2xe^{ax}-ae^{ax}}{2x}=b\Rightarrow\frac{a^2}{2}=b$

Solving simultaneously we obtain $a=3$, $b=\frac{9}{2}$

3 $\lim_{x\to1^+}(\ln(ax))=\lim_{x\to1^-}(\arctan(bx))\Rightarrow\ln(a)=\arctan(b)$

$\lim_{x\to1^+}\frac{1}{x}=\lim_{x\to1^-}\frac{b}{1+b^2x^2}\Rightarrow 1=\frac{b}{1+b^2}\Rightarrow 1+b^2=b$

which has no real solution

Exercise 2G

1 As $\lim_{x\to+\infty}\frac{x^2+2x+1}{e^{3x}+x}=\lim_{x\to+\infty}\frac{2x+2}{3e^{3x}+1}=\lim_{x\to+\infty}\frac{2}{9e^{3x}}=0$ then

$$\lim_{n\to+\infty}\frac{n^2+2n+1}{e^{3n}+n}=0$$

2 As $\lim_{x\to+\infty}\frac{\arctan\left(\frac{1}{x}\right)}{\ln\left(1+\frac{2}{x}\right)}=\lim_{x\to+\infty}\frac{\frac{-\frac{1}{x^2}}{1+\frac{1}{x^2}}}{\frac{-\frac{2}{x^2}}{1+\frac{2}{x}}}=\lim_{x\to+\infty}\frac{1+\frac{2}{x}}{1+\frac{1}{x^2}}=1$

then $\lim_{n\to+\infty}\frac{\arctan\left(\frac{1}{n}\right)}{\ln\left(1+\frac{2}{n}\right)}=1$

3 As $\lim_{x\to+\infty}\left(\frac{x}{2^x}\right)^{\frac{1}{x}}=e^{\lim_{x\to+\infty}\frac{\ln\left(\frac{x}{2^x}\right)}{x}}=e^{\lim_{x\to+\infty}\frac{\ln x-x\ln2}{x}}=e^{\lim_{x\to+\infty}\frac{\frac{1}{x}-\ln2}{1}}$

$=e^{-\ln2}=\frac{1}{2}$ then $\lim_{n\to+\infty}\left(\frac{n}{2^n}\right)^{\frac{1}{n}}=\frac{1}{2}$

Review exercise

1 a Consider the function defined by $f(x)=2\arctan(x)-x$. The solutions of the equation $2\arctan(x)=x$ are the zeros of f. Suppose that $a<b$ are two consecutive zeros of f. Then by Rolle's theorem f' must have at least a zero

on the interval $]a, b[$ (as f clearly satisfies the condition of being continuous and differentiable on any interval of real numbers).

As $f'(x) = \frac{2}{1+x^2} - 1 = 0 \Rightarrow 1 + x^2 = 2 \Rightarrow x = \pm 1$

f can have at most a zero on $]-\infty, -1[$, a zero on $]-1, 1[$ and a zero on $]1, +\infty[$.

b $f(0) = 0$

$\therefore x = 0$ is solution of $2 \arctan x = x$

Therefore the equation has exactly one solution on $]-1, 1[$.

2 a Let $f(x) = \cos(x) - 2x$. Then $f'(x) = \sin(x) - 2$.

$f'(x) = 0 \Rightarrow \sin(x) = -2$ which is impossible.

(1) As f is continuous and differentiable on any interval of real numbers and f' has no zeros, by Rolle's theorem, f has at most one real zero.

(2) As $f(0) = \cos(0) - 2 \times 0 = 1 > 0$ and

$f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) - 2 \times \frac{\pi}{4} = \frac{\sqrt{2}-\pi}{2} < 0$, by

Bolzano's theorem, f has at least a zero on $\left]0, \frac{\pi}{4}\right[$.

By (1) and (2) we can conclude that f has exactly a zero on $\left]0, \frac{\pi}{4}\right[$.

b $\lim_{x\to 0}(\cos(x) - 2x)^{\frac{1}{x}} = e^{\lim_{x\to 0}\frac{\ln(\cos(x)-2x)}{x}}$

$= e^{\lim_{x\to 0}\frac{\frac{-\sin(x)-2}{\cos(x)-2x}}{1}} = e^{-2}$

c $\lim_{x\to +\infty}(\cos(x) - 2x) = -\infty$

3 a Consider the function defined by $f(t) = \sin(t)$.

f is continuous and differentiable in any interval $[x, y]$, $x < y$.

By MTV, applied to any interval $[x, y]$, $x < y$

$\frac{f(x) - f(y)}{x - y} = f'(c)$ where $c \in]x, y[$

Therefore $\left|\frac{f(x)-f(y)}{x-y}\right| = |f'(c)|$

$\Rightarrow |f(x) - f(y)| = |f'(c)| \cdot |x - y|$, $x < y$

As $f'(t) = \cos(t)$ and $|\cos(t)| \le 1$ for any real value of t, $|f(x) - f(y)| = |f'(c)| \cdot |x - y|$

$\Rightarrow |f(x) - f(y)| = |x - y|$. QED

b Consider the function defined by $f(t) = \sqrt{t}$.

f is continuous and differentiable in any interval $[x, y]$, $0 < x < y$.

By MTV, applied to any interval $[x, y]$, $0 < x < y$

$\frac{f(x) - f(y)}{x - y} = f'(c)$ where $c \in]x, y[$

Therefore $\frac{f(y) - f(x)}{y - x} = f'(c)$

$\Rightarrow f(y) - f(x) = f'(c) \cdot (y - x)$, $0 < x < y$

As $f'(t) = \frac{1}{2\sqrt{t}}$ and $\frac{1}{2\sqrt{y}} \le \frac{1}{2\sqrt{t}} \le \frac{1}{2\sqrt{x}}$ for any real value of t, $0 < x \le t \le y$

$\sqrt{y} - \sqrt{x} = \frac{1}{2\sqrt{c}} \cdot (y - x)$

$\Rightarrow -\frac{1}{2\sqrt{y}} \cdot (x - y) \le \sqrt{y} - \sqrt{x} \le -\frac{1}{2\sqrt{x}} \cdot (x - y)$

$\Rightarrow \sqrt{x} - \frac{x-y}{2\sqrt{y}} \le \sqrt{y} \le \sqrt{x} - \frac{x-y}{2\sqrt{x}}$. QED

c Consider the function defined by $f(t) = \ln(x)$.

f is continuous and differentiable in any interval $[x, +\infty]$, $x > 0$.

By MTV, applied to any interval $[x - 1, x]$, $x > 1$

$\frac{f(x) - f(x-1)}{x - (x-1)} = f'(c)$ where $c \in]x-1, x[$

Therefore $\frac{f(x) - f(x-1)}{x - (x-1)} = f'(c)$

$\Rightarrow f(x) - f(x - 1) = f'(c)$, $0 < x - 1 < c < x$

As $f'(t) = \frac{1}{t}$ and $\frac{1}{x} < \frac{1}{t} < \frac{1}{x-1}$ for any real value of t, $0 < x - 1 < t < x$

$\ln(x) - \ln(x - 1) = \frac{1}{t} \Rightarrow \ln\left(\frac{x}{x-1}\right) < \frac{1}{x-1}$,

$x > 1$. QED

d Consider the function defined by $f(t) = \sin(x)$.

f is continuous and differentiable in any interval of real numbers.

By MTV, applied to any interval $[0, x] \subset \left[0, \frac{\pi}{2}\right]$.

$\frac{f(x) - f(0)}{x - 0} = f'(c)$ where $c \in]0, x[\subset \left[0, \frac{\pi}{2}\right]$

Therefore $\frac{f(x) - f(0)}{x - 0} = f'(c)$

$\Rightarrow f(x) - f(0) = f'(c) \cdot x$, $0 < c < x \le \frac{\pi}{2}$

As $f'(t) = \cos(t)$ and $0 < \cos(t) < 1$ for $0 < t < x \le \frac{\pi}{2}$

$\sin(x) - \sin(0) = \cos(c) \cdot x \Rightarrow \sin(x) < x$, $0 < x \le \frac{\pi}{2}$.

As $\sin(0) = 0$, we have $\sin(x) \le x$, for $0 \le x \le \frac{\pi}{2}$. QED

4 $\lim_{x\to 1}\dfrac{x^2-x}{\cot\left(\dfrac{\pi x}{2}\right)}=\lim_{x\to 1}\dfrac{2x-1}{-\dfrac{\pi}{2}\csc^2\left(\dfrac{\pi x}{2}\right)}=\dfrac{1}{-\dfrac{\pi}{2}}=-\dfrac{2}{\pi}$

5 Let $f(x)=\cot\left(\dfrac{\pi x}{2}\right)$, $x\in\left]0,1\right[$.

a $\lim_{x\to 0}(x\cdot f(x))=\lim_{x\to 0}\left(\dfrac{x}{\tan\left(\dfrac{\pi x}{2}\right)}\right)$

$=\lim_{x\to 0}\left(\dfrac{1}{\dfrac{\pi}{2}\sec^2\left(\dfrac{\pi x}{2}\right)}\right)=\dfrac{2}{\pi}$

b $\lim_{x\to 0}\left(\ln(e^x+x)^{f(x)}\right)=\lim_{x\to 0}\dfrac{\ln(e^x+x)}{\tan\left(\dfrac{\pi x}{2}\right)}$

$=\lim_{x\to 0}\dfrac{\dfrac{e^x+1}{e^x+x}}{\dfrac{\pi}{2}\sec^2\left(\dfrac{\pi x}{2}\right)}=\dfrac{2}{\dfrac{\pi}{2}}=\dfrac{4}{\pi}$

6 a i Let f be an even function defined on $[-a,a]$, $a\in\mathbb{R}^+$.

Then $f(-x)=f(x)$ for any $x\in[-a,a]$, $a\in\mathbb{R}^+$. By definition,

$f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\dfrac{f(-x-h)-f(-x)}{-(-h)}$ (f even)

$=\lim_{h\to 0}\dfrac{f(-x-h)-f(-x)}{-(-h)}$

$=-\lim_{-h\to 0}\dfrac{f(-x-h)-f(-x)}{-h}=-f'(-x)$ which shows that f' is odd. QED

ii Let f be an odd function defined on $[-a,a]$, $a\in\mathbb{R}^+$.

Then $f(-x)=-f(x)$ for any $x\in[-a,a]$, $a\in\mathbb{R}^+$. By definition,

$f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$=\lim_{h\to 0}\dfrac{-f(-x-h)+f(-x)}{-(-h)}$ (f odd)

$=\lim_{h\to 0}\dfrac{f(-x-h)-f(-x)}{-h}$

$=\lim_{-h\to 0}\dfrac{f(-x-h)-f(-x)}{-h}=f'(-x)$

which shows that f' is even. QED

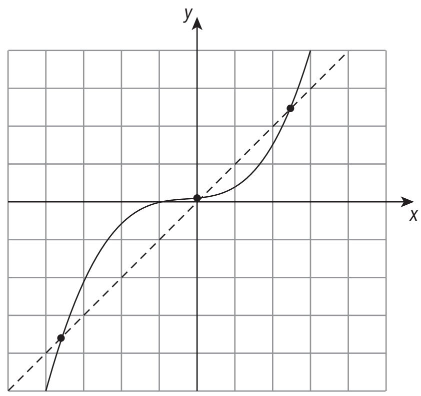
b i Let f be an even non-constant function differentiable (and therefore continuous) on $[-a,a]$, $a\in\mathbb{R}^+$. As $f(-a)=f(a)$ by the Mean value theorem, f' has at least a zero on $\left]-a,a\right[$, $a\in\mathbb{R}^+$ and, by **a**, as f' is odd one of these zeros is at $x=0$ where f' changes sign and therefore f has a maximum or a minimum at $x=0$.

ii Let f be an odd function differentiable (and therefore continuous) on $[-a,a]$, $a\in\mathbb{R}^+$. Then by **a**, as f' is even and $f'(-a)=f'(a)$ by the Mean value theorem, f'' has at least a zero on $\left]-a,a\right[$, $a\in\mathbb{R}^+$ and, by **a**, as f'' is odd one of these zeros is at $x=0$ where f'' changes sign and therefore f has an inflexion point at $x=0$.

c i Let f be an odd function) on $[-a,a]$, $a\in\mathbb{R}^+$. Then the graph of f intersects the line $y=x$ at the origin as $f(0)=0$.

ii Suppose that the graph of intersects the line $y=x$ at $x=b\in\left]0,a\right]$, ie, for $x=b\Rightarrow y=b\Rightarrow f(b)=b$. But then $x=-b\Rightarrow f(-b)=-b\Rightarrow y=-b$ which means that the graph of intersects the line $y=x$ at $x=-b\in\left[-a,0\right[$. Therefore the graph of f intersects the line $y=x$ and odd number of times (at the origin and at pairs of points with coordinates (b,b) and $(-b,-b)$, $b\in\left]0,a\right]$. QED

d e.g.



e Applying the Mean value Theorem to the interval $[-b,0]$, with $b\in\left]0,a\right]$ as in part **c**, we have $f'(c)=\dfrac{f(b)-f(0)}{b-0}=\dfrac{b}{b}=1$ for $c\in\left]-b,0\right[$.

As f is odd the same occurs on the interval $[0,b]$.

3 Modeling dynamic phenomena

Skills check

1 a $x\,e^{y} + x^{2} = 1 \Rightarrow x\,e^{y} = 1 - x^{2} \Rightarrow e^{y}$

$$= \frac{1-x^2}{x} \Rightarrow y = \ln\left(\frac{1-x^2}{x}\right)$$

b $\ln\left(\frac{x+1}{y}\right) = 2x \Rightarrow \frac{x+1}{y} = e^{2x} \Rightarrow y = \frac{x+1}{e^{2x}}$

or $y = (x+1)e^{-2x}$

c $\tan\left(\frac{x}{y}\right) = 1 \Rightarrow \frac{x}{y} = \frac{\pi}{4} \Rightarrow y = \frac{4x}{\pi}$

(assuming principal restriction)

d $\arccos(xy - 1) = 2x \Rightarrow xy - 1 = \cos(2x)$

$$\Rightarrow y = \frac{\cos 2x + 1}{x}$$

2 a $\int x^2 e^x dx = x^2 e^x - 2\int x\,e^x dx$

$= x^2 e^x - 2(x\,e^x - \int e^x dx)$

i $x^2 e^x - 2x\,e^x + 2e^x + c$ or $(x^2 - 2x + 2)e^x + c$

ii $\int \sin(2x)\cdot e^x dx = \sin(2x)\cdot e^x - 2\int \cos(2x)\cdot e^x dx$

$\Rightarrow \int \sin(2x)\cdot e^x dx$

$= \sin(2x)\cdot e^x - 2(\cos(2x)\cdot e^x - 2\int \sin(2x)\cdot e^x dx)$

$\Rightarrow 5\int \sin(2x)\cdot e^x dx = (\sin(2x) - 2\cos(2x))\cdot e^x + c$

$\Rightarrow \int \sin(2x)\cdot e^x dx = \frac{1}{5}(\sin(2x) - 2\cos(2x))\cdot e^x + c$

iii $\int \frac{e^x}{e^x - 1}dx = \int \frac{1}{u}du$ where $u = e^x - 1 \Rightarrow du = e^x dx$

As $\int \frac{1}{u}du = \ln|u| + c$ then $\int \frac{e^x}{e^x-1}dx = \ln|e^x - 1| + c$

iv $\int \frac{\cos(x)}{\sin^2(x)}dx = \int \frac{1}{u^2}du$ where $u = \sin(x)$

$\Rightarrow du = \cos(x)\,dx$

As $\int \frac{1}{u^2}du = -\frac{1}{u} + c$ then $\int \frac{\cos(x)}{\sin^2(x)}dx = -\frac{1}{\sin(x)} + c.$

b i $$y' = \frac{\frac{1\cdot(x-1) - x\cdot 1}{(x-1)^2}}{\sqrt{1-\left(\frac{x}{x-1}\right)^2}} = \frac{\frac{-1}{(x-1)^2}}{\sqrt{\frac{(x-1)^2 - x^2}{(x-1)^2}}} = \frac{\frac{-1}{(x-1)^2}}{\sqrt{\frac{-2x+1}{(x-1)^2}}}$$

$$= -\frac{1}{|x-1|\sqrt{1-2x}}$$

ii $\frac{\sin(2x)}{\sin^2(x)+1}$

3 a $\frac{dy}{dx}(x=1) = e^2 - \ln(1) = e^2$

b $\frac{dy}{dx}\left(x = \frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$

Exercise 3A

1

EQUATION	LINEAR	ORDER	COEFFICIENTS
1	YES	3	Constant
2	NO	1	Constant
3	YES	2	Variable
4	NO	1	Constant
5	YES	1	Variable
6	YES	1	Variable
7	NO	1	Variable

2 a $y = e^{-x} \Rightarrow y' = -e^{-x}$

Substitute into $y' + y = 0$ to obtain $e^{-x} - e^{-x} = 0$ which is true for all $x \in \mathbb{R}$

$\therefore y = e^{-x}$ is solution of the differential equation $y' + y = 0$.

b $y = \sin(x) \Rightarrow y' = \cos(x) \Rightarrow y'' = -\sin(x)$

Substitute into $y'' + y = 0$ to obtain $-\sin(x) + \sin(x) = 0$ which is true for all $x \in \mathbb{R}$

$\therefore y = \sin(x)$ is solution of the differential equation $y'' + y = 0$.

c $y = x\,e^{-x} \Rightarrow y' = e^{-x} - x\,e^{-x}$

Substitute into $y' + y = e^{-x}$ to obtain $e^{-x} - e^{-x} = 0$ which is true for all $x \in \mathbb{R}$

$\therefore y = x\,e^{-x}$ is solution of the differential equation $y' + y = e^{-x}$.

3 a Differentiating implicitly $e^{xy} + x + y = 0$,

$$y\,e^{xy} + x\,e^{xy}\frac{dy}{dx} + 1 + \frac{dy}{dx} = 0$$

$$\Rightarrow (1 + x\,e^{xy})\frac{dy}{dx} = -(1 + y\,e^{xy})$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1 + y\,e^{xy}}{1 + x\,e^{xy}} \text{ QED}$$

b Differentiating implicitly $x^2 + y^2 = r^2$, r constant

$2x + 2y\frac{dy}{dx} = 0 \Rightarrow y\frac{dy}{dx} = -x \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$ QED

Investigation

1 For example, $y = e^x \Rightarrow y' = e^x$.
Substitute into $y' - y = 0$ to obtain $e^x - e^x = 0$ which is true for all $x \in \mathbb{R}$ $\therefore y = e^x$ is solution of the differential equation $y' - y = 0$
For example, $y = 2e^x \Rightarrow y' = 2e^x$.
Substitute into $y' - y = 0$ to obtain $2e^x - 2e^x = 0$ which is true for all $x \in \mathbb{R}$
$\therefore y = 2e^x$ is solution of the differential equation $y' - y = 0$
In general, $y = Ae^x \Rightarrow y' = Ae^x$.
Substitute into $y' - y = 0$ to obtain $Ae^x - Ae^x = 0$ which is true for all $x \in \mathbb{R}$
$\therefore y = Ae^x$ is a general solution of the differential equation $y' - y = 0$.
When $x = 0 \Rightarrow y = 3 \Rightarrow Ae^0 = 3 \Rightarrow A = 3$.
$\therefore y = 3e^x$ is the particular solution (curve) that contains the point (0, 3).

2 For example, $y = -\frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{1}{x^2} \Rightarrow \frac{dy}{dx} = y^2$.

For example, $y = -\frac{1}{x-1} \Rightarrow \frac{dy}{dx} = \frac{1}{(x-1)^2} \Rightarrow \frac{dy}{dx} = y^2$

In general, $y = -\frac{1}{x+k} \Rightarrow \frac{dy}{dx} = \frac{1}{(x+k)^2} \Rightarrow \frac{dy}{dx} = y^2$.

When $x = 1 \Rightarrow y = 1 \Rightarrow 1 = -\frac{1}{1+k} \Rightarrow k = -2$.

$\therefore y = -\frac{1}{x-2}$ is the particular solution (curve) that contains the point (1, 1).

3 No solution as $(y'(x))^2 \geq 0$

Exercise 3B

1 **a** $\frac{dx}{dt} = t + t^2 \Rightarrow x = \frac{t^2}{2} + \frac{t^3}{2} + c$

b $\frac{dy}{dx} = \frac{1}{x} \Rightarrow y = \ln|x| + c$

c $\frac{dz}{dx} = x\cos(x) \Rightarrow z = x\sin(x) - \int \sin(x)dx$

$\Rightarrow z = x\sin(x) + \cos(x) + c$

d $\frac{dw}{dx} = x\,e^{2x} \Rightarrow w = \frac{1}{2}x\,e^{2x} - \frac{1}{2}\int e^{2x}dx$

$\Rightarrow w = \frac{1}{2}x\,e^{2x} - \frac{1}{4}e^{2x} + c$

2 **a** $\frac{dy}{dx} = 3x + x^2 \Rightarrow y = \frac{3}{2}x^2 + \frac{1}{3}x^3 + c$

$y(1) = 4 \Rightarrow \frac{3}{2} + \frac{1}{3} + c = 4 \Rightarrow c = \frac{13}{6}$

$\therefore y = \frac{3}{2}x^2 + \frac{1}{3}x^3 + \frac{13}{6}$

b $\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}} \Rightarrow y = \frac{1}{2}\int \frac{-2x}{\sqrt{4-x^2}}dx$

$u = 4 - x^2 \Rightarrow du = -2x\,dx$

$y = \frac{1}{2}\int \frac{1}{\sqrt{u}}du \Rightarrow y = \sqrt{u} + c$

$\therefore y = \sqrt{4-x^2} + c$

$y(0) = 1 \Rightarrow \sqrt{4} + c = 1 \Rightarrow c = -1$

$\therefore y = \sqrt{4-x^2} - 1$

3 $\frac{dy}{dx} = \frac{1}{x+2} - \frac{1}{2}\sin(x) \Rightarrow y = \int \frac{1}{x+2}dx - \frac{1}{2}\int \sin(x)dx$

$y = \ln|x+2| + \frac{1}{2}\cos(x) + c$

$y(0) = 2 \Rightarrow \ln 2 + \frac{1}{2}\cos(0) + c = 2$

$\Rightarrow c = \frac{3}{2} - \ln 2$

$\therefore y = \ln|x+2| + \frac{1}{2}\cos(x) + \frac{3}{2} - \ln 2$

4 $y = \sin(kx) - kx\cos(kx)$

$\Rightarrow \frac{dy}{dx} = k\cos(kx) - k\cos(kx) + k^2x\sin(kx)$

$\Rightarrow \frac{dy}{dx} = k^2x\sin(kx)$ QED

5 $\frac{dy}{dx} = e^{-2x} - \frac{1}{x-1} \Rightarrow y = \int\left(e^{-2x} - \frac{1}{x-1}\right)dx$

$\therefore y = -\frac{1}{2}e^{-2x} - \ln|x-1| + c$

$y(0) = 1 \Rightarrow -\frac{1}{2} - \ln 1 + c = 1 \Rightarrow c = \frac{3}{2}$

$\therefore y = -\frac{1}{2}e^{-2x} - \ln|x-1| + \frac{3}{2}$

6 **a** $\frac{dV}{dt} = 8 \Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 8$

$\Rightarrow 4\pi r^2\frac{dr}{dt} = 8\left(\Rightarrow \frac{dr}{dt} = \frac{2}{\pi r^2}\right)$

b $\frac{dr}{dt} = \frac{2}{\pi r^2} \Rightarrow \int r^2 dr = \frac{2}{\pi}\int dt \Rightarrow \frac{1}{3}r^3 = \frac{2}{\pi}t + c$

$r(1) = 5 \Rightarrow \frac{1}{3} \times 125 = \frac{2}{\pi} + c \Rightarrow c = \frac{125}{3} - \frac{2}{\pi}$

$\therefore \frac{1}{3}r^3 = \frac{2}{\pi}t + \frac{125}{3} - \frac{2}{\pi} \Rightarrow r = \sqrt[3]{\frac{6}{\pi}(t-1) + 125}$

7 $v = \frac{1}{2+t^2} \Rightarrow \frac{ds}{dt} = \frac{1}{2+t^2} \Rightarrow s = \int \frac{1}{2+t^2}dt$

$\therefore s = \frac{1}{\sqrt{2}}\arctan\left(\frac{t}{\sqrt{2}}\right) + c$

Exercise 3C

1 **a** $\frac{dy}{dx} = y - 3 \Rightarrow \frac{dy}{y-3} = dx$

$\Rightarrow \int \frac{1}{y-3} dy = \int dx \Rightarrow \ln|y-3| = x + c$

$|y-3| = e^{x+c} \Rightarrow y = Ae^x + 3$

b $\frac{dy}{dx} = \frac{1}{3y^2 + y - 1} \Rightarrow (3y^2 + y - 1)dy = dx$

$\Rightarrow \int (3y^2 + y - 1)dy$

$= \int dx \Rightarrow y^3 + \frac{y^2}{2} - y = x + c$

c $\frac{dy}{dx} = y^2 + 4 \Rightarrow \frac{dy}{y^2+4} = dx \Rightarrow \int \frac{dy}{y^2+4} = \int dx$

$\therefore \frac{1}{2} \arctan\left(\frac{y}{2}\right) = x + c$

d $\frac{dy}{dx} = 2y + 1 \Rightarrow \frac{dy}{2y+1} = dx \Rightarrow \int \frac{1}{2y+1} dy = \int dx$

$\Rightarrow \frac{1}{2} \int \frac{2}{2y+1} dy = \int dx$

$\therefore \frac{1}{2} \ln|2y+1| = x + c$

e $\frac{dy}{dx} = \cos^2(3y-1) \Rightarrow \frac{dy}{\cos^2(3y-1)} = dx$

$\Rightarrow \int \frac{dy}{\cos^2(3y-1)} = \int dx$

$\Rightarrow \frac{1}{3} \int \frac{3dy}{\cos^2(3y-1)} = \int dx$

$\therefore \frac{1}{3} \arctan(3y-1) = x + c$

f $\frac{dy}{dx} = \frac{1+\sin^2 y}{\cos y} \Rightarrow \frac{\cos y}{1+\sin^2 y} dy = dx$

$\Rightarrow \int \frac{\cos y}{1+\sin^2 y} dy = \int dx$

$\therefore \arctan(\sin y) = x + c$

2 **a** $e^{-x} \frac{dy}{dx} = \frac{3}{y} \Rightarrow \int y dy = 3 \int e^x dx \Rightarrow \frac{y^2}{2} = 3e^x + c$

To check answers, differentiate both sides of the equation, e.g.

$\frac{d}{dx}\left(\frac{y^2}{2}\right) = \frac{d}{dx}(3e^x + c) \Rightarrow y \frac{dy}{dx} = 3e^x$

$\Rightarrow e^{-x} \frac{dy}{dx} = \frac{3}{y}$

b $\frac{5}{y-1} \frac{dy}{dx} = \frac{2}{x} \Rightarrow \frac{5}{y-1} dy = \frac{2}{x} dx$

$\Rightarrow \int \frac{5}{y-1} dy = \int \frac{2}{x} dx$

$\therefore 5\ln|y-1| = 2\ln|x| + c \Rightarrow (y-1)^5 = Ax^2$

c $\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{1}{y} dy = \frac{1}{x} dx \Rightarrow \int \frac{1}{y} dy = \int \frac{1}{x} dx$

$\therefore \ln|y| = \ln|x| + c \Rightarrow y = Ax$

d $\frac{dy}{dx} = -\frac{y^2+1}{x^2+1} \Rightarrow \frac{1}{y^2+1} dy = -\frac{1}{x^2+1} dx$

$\Rightarrow \int \frac{1}{y^2+1} dy = -\int \frac{1}{x^2+1} dx$

$\therefore \arctan(y) = -\arctan(x) + c$

or $y = -\tan(\arctan(x) + c)$

e $e^{-x} \frac{dy}{dx} = \frac{2x}{y} \Rightarrow \int y dy = 2 \int x e^x dx$

$\Rightarrow \frac{y^2}{2} = 2x e^x - 2 \int e^x dx$

$\therefore \frac{y^2}{2} = 2x e^x - 2e^x + c \Rightarrow y^2 = 4(x-1)e^x + c$

f $\frac{x}{y^2+1} dy = \frac{2}{x} dx \Rightarrow \int \frac{1}{y^2+1} dy = -2 \int -\frac{1}{x^2} dx$

$\therefore \arctan(y) = -\frac{2}{x} + c$

3 **a** $(1 + \cos(2x)) \frac{dy}{dx} = 2\sin(2x) y \Rightarrow \frac{1}{y} dy$

$= \frac{2\sin(2x)}{1+\cos(2x)} dx$

$\int \frac{dy}{y} = \int \frac{2\sin 2x}{1+\cos 2x} dx$

$\Rightarrow \ln|y| = -\ln|1+\cos 2x| + c$

$\Rightarrow y = \frac{c}{1+\cos 2x}$

$y(0) = 1 \Rightarrow c = 2$

$y = \frac{2}{1+\cos 2x} \Rightarrow y = \frac{2}{1+(2\cos^2 x - 1)}$

$\therefore y = \sec^2 x$

b $(1+x^2) \frac{dy}{dx} = 1 + y^2 \Rightarrow \frac{1}{1+y^2} dy = \frac{1}{1+x^2} dx$

$\Rightarrow \int \frac{1}{1+y^2} dy = \int \frac{1}{1+x^2} dx$

$\Rightarrow \arctan(y) = \arctan(x) + c$

$y(0) = 2 \Rightarrow \arctan(2) = \arctan(0) + c$

$\therefore c = \arctan(2)$

$\therefore y = \tan(\arctan(x) + \arctan(2))$

$\therefore y = \frac{x+2}{1-2x}$

c $e^x \dfrac{dy}{dx} + 1 = y \Rightarrow \dfrac{1}{y-1}\,dy = e^{-x}dx$

$\Rightarrow \displaystyle\int \frac{1}{y-1}\,dy = \int e^{-x}dx$

$\therefore \ln|y-1| = -e^{-x} + c$

$y(0) = 2 \Rightarrow \ln 1 = -e^0 + c$

$\therefore c = 1$

$\therefore \ln|y-1| = 1 - e^{-x}$

4 **a** $\dfrac{dV}{dt} = 8 \Rightarrow \dfrac{d}{dt}\left(\dfrac{4}{3}\pi r^3\right) = 8$

$\Rightarrow 4\pi r^2 \dfrac{dr}{dt} = 8 \left(\Rightarrow \dfrac{dr}{dt} = \dfrac{2}{\pi r^2}\right)$

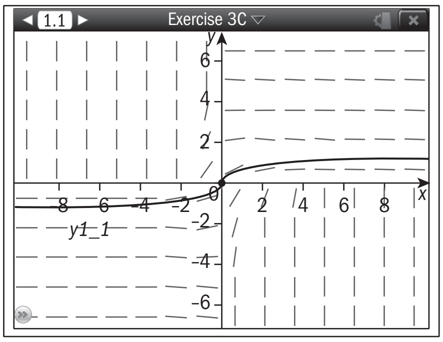
b $\dfrac{dr}{dt} = \dfrac{2}{\pi r^2} \Rightarrow \displaystyle\int r^2 dr = \frac{2}{\pi}\int dt \Rightarrow \frac{1}{3}r^3 = \frac{2}{\pi}t + c$

$r(1) = 5 \Rightarrow \dfrac{1}{3} \times 125 = \dfrac{2}{\pi} + c \Rightarrow c = \dfrac{125}{3} - \dfrac{2}{\pi}$

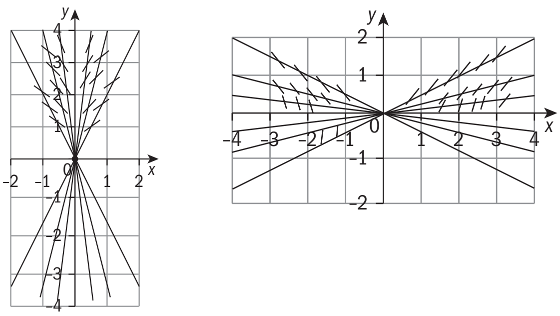
$\therefore \dfrac{1}{3}r^3 = \dfrac{2}{\pi}t + \dfrac{125}{3} - \dfrac{2}{\pi} \Rightarrow r = \sqrt[3]{\dfrac{6}{\pi}(t-1) + 125}$

5 **a** $\dfrac{dy}{dx} = e^{-xy}$ is not a variable separable equation because it cannot be written in the form $\dfrac{dy}{dx} = \dfrac{f(x)}{g(y)}$

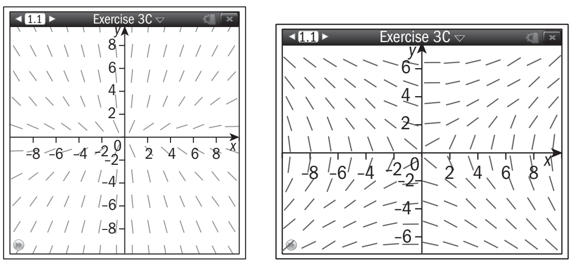
b and c



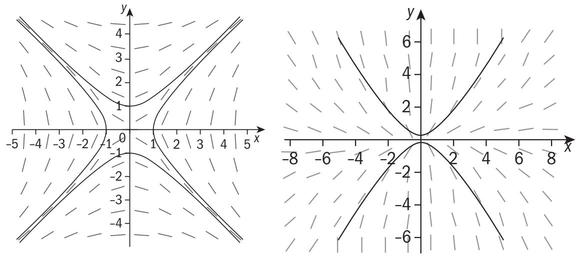
6 **a**



b



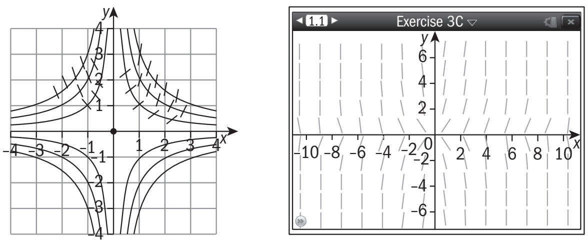
c



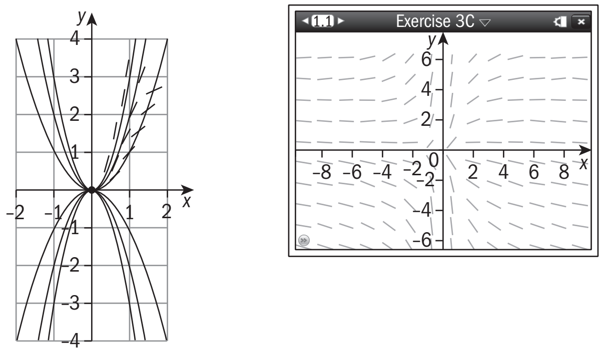
d At each point (x, y), $x \neq 0$, $y \neq 0$ the gradients of the tangents to the solutions of each DE are given by $\dfrac{dy}{dx} = \dfrac{2y}{x}$ and $\dfrac{dy}{dx} = -\dfrac{x}{2y}$.

As $\left(\dfrac{2y}{x}\right) \times \left(-\dfrac{x}{2y}\right) = -1$ these tangents are orthogonal at these points.

7 **a**



b



Short investigation

year	annually	monthly	daily	hourly	continously
40	4,115.19	4211.61	4220.40	4220.68	4220.70

Exercise 3D

1 The model is $Q(t) = Q_0 e^{-\frac{\ln 2}{5568}t}$

$Q(t) = 0.55 \cdot Q_0 \Rightarrow Q_0 e^{-\frac{\ln 2}{5568}t}$

$= 0.55 \cdot Q_0 \Rightarrow -\dfrac{\ln 2}{5568}t = 0.55$

$\Rightarrow t = 4800$ years

2 The model is $Q(t) = Q_0 e^{-\frac{\ln 2}{5530}t}$

$Q(t) = 0.55 \cdot Q_0 \Rightarrow Q_0 e^{-\frac{\ln 2}{5530}t}$

$= 0.55 \cdot Q_0 \Rightarrow -\dfrac{\ln 2}{5530}t = 0.55$

$\Rightarrow t = 4770$ years

3 **a** $\frac{dV}{dt} = -0.00015V$

b $\frac{dV}{dt} = -0.00015V \Rightarrow \int \frac{dV}{R} = -0.00015 \int dt$

$\Rightarrow \ln V = -0.00015t + c$

$\therefore V = V_0 e^{-0.00015t}$

$V(2) = 0.0008 \Rightarrow V_0 e^{-0.00015\times 2} = 0.0008$

$\Rightarrow V_0 = 0.00080024... < 0.001$

So, the driver's blood alcohol rate was within the permissible limits.

4 $\frac{dP}{dt} = 0.0004P(250-P) \Rightarrow \frac{dP}{P(250-P)} = \frac{1}{2500}dt$

$\int \frac{dP}{P(250-P)} = \int \frac{1}{2500}dt$

$\Rightarrow \frac{1}{250}\int\left(\frac{1}{P} + \frac{1}{250-P}\right)dP = \frac{1}{2500}\int dt$

$\int\left(\frac{1}{P} + \frac{1}{250-P}\right)dP = \frac{1}{10}\int dt \Rightarrow \ln\left|\frac{P}{250-P}\right| = \frac{1}{10}t + c$

$\left|\frac{P}{250-P}\right| = Ae^{0.1t}$

$P(0) = 35 \Rightarrow A = \frac{7}{43}$

$\therefore \left|\frac{P}{250-P}\right| = \frac{7}{43}e^{0.1t}$ and $P(t) = 240 \Rightarrow t = 49.93...,$

i.e., approx. 50 years.

5 **a** $\frac{dP}{dt} = \frac{0.03}{1+t}P \Rightarrow \frac{dP}{P} = \frac{0.03}{1+t}dt \Rightarrow P = A(1+t)^{0.03}$

$P(0) = 100 \Rightarrow A = 100$

$\therefore P = 100(1+t)^{0.03}$ **and** $P(10) = 100 \times (11)^{0.03}$

$\Rightarrow P(10) = 107.459...$

$\therefore$ 107.46 euros

b $P = 100(1+t)^{I_0}$

$P(10) = 100 \times 11^{I_0}$

c $\frac{dP}{dt} = (0.01 + 0.001t)P \Rightarrow \frac{dP}{P} = (0.01 + 0.001t)dt$

$\int \frac{dP}{P} = \int (0.01 + 0.001t)dt$

$\Rightarrow \ln|P| = 0.01t + 0.0005t^2 + c$

$P(0) = 100 \Rightarrow c = \ln(100)$

$P = 100e^{0.01t + 0.0005t^2}$

$P(10) = 116.183... = 116.18$ euros

6 **a** $m\frac{dv}{dt} = mg - kv^2 \Rightarrow \frac{dv}{dt} = g - \frac{k}{m}v^2$

$\Rightarrow g\frac{dv}{g^2 - \frac{gk}{m}v^2} = dt$

$\alpha = \sqrt{\frac{gk}{m}} \Rightarrow \frac{g}{g^2 - \alpha^2 v^2}dv = dt$

As $\frac{g}{g^2 - \alpha^2 v^2} = \frac{1}{2}\left(\frac{1}{g - \alpha v} + \frac{1}{g + \alpha v}\right)$ the equation can be written as

$\left(\frac{1}{g - \alpha v} + \frac{1}{g + \alpha v}\right)dv = 2dt$

b $\int \frac{g}{g^2 - \alpha^2 v^2}dv = \int dt \Rightarrow \int\left(\frac{1}{g - \alpha v} + \frac{1}{g + \alpha v}\right)dv$

$= 2\int dt - \frac{1}{\alpha}\ln\left|\frac{g + \alpha v}{g - \alpha v}\right| = t^2 + c \Rightarrow \frac{g + \alpha v}{g - \alpha v} = Ae^{-\frac{1}{\alpha}t^2}$

$v = \frac{g}{\alpha}\frac{Ae^{-\frac{1}{\alpha}t^2} - 1}{Ae^{-\frac{1}{\alpha}t^2} + 1}$

$\lim_{x\to\infty} v(t) = \frac{g}{\alpha}$

Exercise 3E

1 **a** $\frac{dy}{dx} + \underbrace{3}_{p(x)} y = \underbrace{e^{2x}}_{q(x)}$

$I(x) = e^{\int 3dx} = e^{3x}$

Multiplying both sides of $\frac{dy}{dx} + 3y = e^x$ by $I(x)$,

$e^x \cdot \frac{dy}{dx} + 3e^{3x}y = e^{5x} \Rightarrow \frac{d}{dx}(e^{3x}\cdot y) = e^{5x}$

which is an exact DE

$\therefore e^{3x}\cdot y = \int e^{5x}dx \Rightarrow e^{3x}\cdot y = \frac{1}{5}e^{5x} + c$

$\therefore y = \frac{1}{5}e^{2x} + c\cdot e^{-3x}$

b $\frac{dy}{dx} + \underbrace{(2x+1)}_{p(x)} y = \underbrace{e^{-x^2}}_{q(x)}$

$I(x) = e^{\int(2x+1)dx} = e^{x^2+x}$

Multiplying both sides of $\frac{dy}{dx} + (2x+1)y = e^{-x^2}$ by $I(x)$,

$e^{-x^2}\cdot\frac{dy}{dx} + (2x+1)e^{x^2+x}y = e^x \Rightarrow \frac{d}{dx}(e^{x^2+x}\cdot y) = e^x$

which is an exact DE

$\therefore e^{x^2+x}\cdot y = \int e^x dx \Rightarrow e^{x^2+x}\cdot y = e^x + c$

$\therefore y = e^{-x^2} + c\cdot e^{-x^2-x}$

c $\frac{dy}{dx} + \underbrace{1}_{p(x)} \cdot y = \underbrace{e^{-2x}\cos(x)}_{q(x)}$

$I(x) = e^{\int 1dx} = e^x$

Multiplying both sides of $\frac{dy}{dx} + y = e^{-2x}\cos(x)$ by $I(x)$,

$e^x\frac{dy}{dx} + e^x y = e^{-x}\cos(x) \Rightarrow \frac{d}{dx}(e^x \cdot y) = e^{-x}\cos(x)$

which is an exact DE

$\therefore e^x \cdot y = \int e^{-x}\cos(x)dx$

$\Rightarrow e^x \cdot y = \frac{1}{2}e^{-x}(\sin(x) - \cos(x)) + c$

(use integration by parts)

$\therefore y = \frac{1}{2}e^{-2x}(\sin(x) - \cos(x)) + c \cdot e^{-x}$

$y = \frac{1}{2}e^{-x^2}(\sin x - \cos x) + Ae^{-x}$

d $y = \left(\frac{x^2}{2} + c\right)\cos x$

$\frac{dy}{dx} + \underbrace{\tan(x)}_{p(x)} \cdot y = \underbrace{\cos(x)}_{q(x)}$

$I(x) = e^{\int \tan(x)dx} = e^{-\ln(\cos(x))} = \sec(x)$

Multiplying both sides of $\frac{dy}{dx} + y\tan(x) = \cos(x)$ by $I(x)$,

$\sec(x) \cdot \frac{dy}{dx} + \tan(x)\sec(x) \cdot y = 1 \Rightarrow \frac{d}{dx}$

$(\sec(x) \cdot y) = 1$ which is an exact DE

$\therefore \sec(x) \cdot y = \int 1dx \Rightarrow \sec(x) \cdot y = x + c$ (use integration by parts)

$\therefore y = (x + c) \cdot \cos(x)$

e $\frac{dy}{dx} + \underbrace{\cot(x)}_{p(x)} \cdot y = \underbrace{\cos(x)}_{q(x)}$

$I(x) = e^{\int \cot(x)dx} = e^{\ln(\sin(x))} = \sin(x)$

Multiplying both sides of $\frac{dy}{dx} + y\cot(x) = \cos(x)$ by $I(x)$,

$\sin(x)\frac{dy}{dx} + \cos(x) \cdot y = \sin(x)$

$\Rightarrow \frac{d}{dx}(\sin(x) \cdot y) = \sin(x)$ which is an exact DE

$\therefore \sin(x) \cdot y = \int \underbrace{\sin(x)\cos(x)}_{\frac{1}{2}\sin(2x)}dx$

$\Rightarrow \sin(x) \cdot y = -\frac{1}{4}\cos(2x) + c$

$\therefore y = \left(-\frac{1}{4}\cos(2x) + c\right) \cdot \csc(x)$

2 a $\frac{dy}{dx} = x - y \Rightarrow \frac{dy}{dx} + y = x$

$I(x) = e^{\int 1dx} = e^x$

Multiplying both sides of $\frac{dy}{dx} + y = c$ by $I(x)$,

$e^x\frac{dy}{dx} + e^x y = x \cdot e^x \Rightarrow \frac{d}{dx}(e^x \cdot y) = x \cdot e^x$ which is an exact DE

$\therefore e^x \cdot y = \int x \cdot e^x dx \Rightarrow e^x \cdot y = (x - 1)e^x + c$ (use integration by parts)

$\therefore y = x - 1 + c \cdot e^{-x}$

$y(0) = -2 \Rightarrow -1 + c = -2 \Rightarrow c = -1$

$\therefore y = x - 1 - e^{-x}$

b $\frac{dy}{dx} - 2y = \sin x$

$I(x) = e^{\int -2dx} = e^{-2x}$

Multiplying both sides of $\frac{dy}{dx} - 2y = \sin x$ by $I(x)$,

$e^{-2x}\frac{dy}{dx} - 2e^{-2x}y = e^{-2x}\sin(x)$

$\Rightarrow \frac{d}{dx}(e^{-2x}y) = e^{-2x}\sin(x)$

which is an exact DE

$\therefore e^{-2x}\ y = \int e^{-2x}\sin(x)dx$

$\Rightarrow e^{-2x}y = \frac{1}{5}(-2\sin(x) - \cos(x))\ e^{-2x} + c$

$y(0) = 0 \Rightarrow c = \frac{1}{5}$

$y = \frac{1}{5}(-2\sin(x) - \cos(x) + e^{2x})$

c $\frac{dy}{dx} + (x + 1)\ y = e^{-\frac{x^2}{2}}$

$I(x) = e^{\int (x+1)dx} = e^{\frac{x^2}{2}+x}$

Multiplying both sides of $\frac{dy}{dx} + (x + 1)\ y = e^{-\frac{x^2}{2}}$ by $I(x)$,

$e^{\frac{x^2}{2}+x}\frac{dy}{dx} + e^{\frac{x^2}{2}+x}(x + 1)\ y = e^x \Rightarrow \frac{d}{dx}\left(e^{\frac{x^2}{2}+x}y\right) = e^x$

which is an exact DE

$\therefore e^{\frac{x^2}{2}+x}y = \int e^x dx \Rightarrow e^{\frac{x^2}{2}+x}y = e^x + c$

$y(0) = 2 \Rightarrow c = 1$

$y = e^{-\frac{x^2}{2}} + e^{-\frac{x^2}{2}-x}$

3 a $x\frac{dy}{dx}+y=x^2+x \Rightarrow \frac{d}{dx}(xy)=x^2+x$

$\therefore$ the equation is exact.

$\frac{d}{dx}(xy)=x^2+x \Rightarrow xy=\int(x^2+x)\,dx$

$\Rightarrow xy=\frac{x^3}{3}+\frac{x^2}{2}+c$

$\therefore y=\frac{x^2}{3}+\frac{x}{2}+\frac{c}{x}$

$x\frac{dy}{dx}+y=x^2+x \Rightarrow \frac{dy}{dx}+\underbrace{\frac{1}{x}}_{p(x)}\cdot y=\underbrace{x+1}_{q(x)}$

$\therefore$ the equation is linear

b $(x+1)\frac{dy}{dx}+y=\sin(x)e^x \Rightarrow \frac{d}{dx}((x+1)y)=\sin(x)e^x$

$\therefore$ the equation is exact.

$\frac{d}{dx}((x+1)y)=\sin(x)e^x \Rightarrow (x+1)y=\int\sin(x)\,e^x dx$

$\Rightarrow (x+1)y=\frac{1}{2}(\sin x-\cos x)\,e^x+c$

$\therefore y=\frac{\sin x-\cos x}{2(x+1)}e^x+\frac{c}{x+1}$

$(x+1)\frac{dy}{dx}+y=\sin(x)e^x \Rightarrow \frac{dy}{dx}+\underbrace{\frac{1}{x+1}}_{p(x)}\cdot y=\underbrace{\frac{\sin(x)e^x}{x+1}}_{q(x)}$

$\therefore$ the equation is linear

c $(\cos x)\frac{dy}{dx}-(\sin x)y=\cos^4 x$

$\Rightarrow \frac{d}{dx}(\cos(x)\ y)=\cos^4 x$

$\therefore$ the equation is exact.

$\frac{d}{dx}(\cos(x)\ y)=\cos^4 x \Rightarrow \cos(x)\ y=\int\cos^4 x\ dx$

As $\cos^4 x=\left(\frac{1+\cos(2x)}{2}\right)^2=\frac{1+2\cos(2x)+\cos^2(2x)}{4}$

$=\frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{8}(1+\cos(4x))$

$=\frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)$

$\Rightarrow \cos(x)y=\int\frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\,dx$

$\Rightarrow \cos(x)y=\frac{3}{8}x+\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+c$

$\therefore y=\left(\frac{3}{8}x+\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+c\right)\sec(x)$

$(\cos x)\frac{dy}{dx}-(\sin x)y=\cos^4 x \Rightarrow \frac{dy}{dx}-\underbrace{\tan(x)}_{p(x)}\cdot y=\underbrace{\cos^3(x)}_{q(x)}$

$\therefore$ the equation is linear

d $(\sin(x))\frac{dy}{dx}+(\cos(x))y=\tan(x)$

$\Rightarrow \frac{d}{dx}(\sin(x)y)=\tan(x)$

$\therefore$ the equation is exact.

$\frac{d}{dx}(\sin(x)y)=\tan(x) \Rightarrow \sin(x)y=\int\tan x\ dx$

$\Rightarrow \sin(x)y==\int\tan(x)\,dx$

$\Rightarrow \sin(x)y=-\ln|\cos x|+c$

$\therefore y=\frac{c-\ln|\cos x|}{\sin x}$ or $y=\left(\ln|\sec(x)|+c\right)\csc(x)$

$(\sin(x))\frac{dy}{dx}+(\cos(x))\ y=\tan(x)$

$\Rightarrow \frac{dy}{dx}+\underbrace{\cot(x)}_{p(x)}\cdot y=\underbrace{\sec(x)}_{q(x)}$

$\therefore$ the equation is linear

4 a $x^2\frac{dy}{dx}-x^2y=y \Rightarrow \frac{dy}{dx}-\left(1+x^{-2}\right)y=0$

$\therefore$ the equation is linear

$I(x)=e^{\int -1-x^{-2}dx}=e^{-x+\frac{1}{x}}$

Multiplying both sides of $\frac{dy}{dx}-(1+x^{-2})y=0$ by $I(x)$,

$e^{-x+\frac{1}{x}}\frac{dy}{dx}-(1+x^{-2})\,e^{-x+\frac{1}{x}}y=0 \Rightarrow \frac{d}{dx}\left(e^{-x+\frac{1}{x}}y\right)=0$

which is an exact DE

$e^{-x+\frac{1}{x}}y=A \Rightarrow y=Ae^{-\frac{1}{x}+x}$

$\frac{dy}{dx}-y=x^{-2}y \Rightarrow \frac{dy}{dx}=(x^{-2}+1)y$

$\Rightarrow \frac{dy}{y}=(x^{-2}+1)dx$

$\therefore$ the equation is variable separable

$\int\frac{dy}{y}=\int(x^{-2}+1)\,dx \Rightarrow \ln|y|=-\frac{1}{x}+x+c$

$\therefore y=Ae^{-\frac{1}{x}+x}$

b $(x^2+1)\frac{dy}{dx}-xy=0 \Rightarrow \frac{dy}{dx}-\frac{x}{x^2+1}\ y=0$

$\therefore$ the equation is linear

$I(x)=e^{\int -\frac{x}{x^2+1}dx}=e^{-\frac{1}{2}\ln(x^2+1)}=\frac{1}{\sqrt{x^2+1}}$

Multiplying both sides of $\frac{dy}{dx}-\frac{x}{x^2+1}y=0$ by $I(x)$,

$\frac{1}{\sqrt{x^2+1}}\frac{dy}{dx}-\frac{x}{x^2+1}\frac{1}{\sqrt{x^2+1}}\ y=0$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{\sqrt{x^2+1}}\ y\right)=0$

which is an exact DE

$$\frac{1}{\sqrt{x^2+1}}\ y = 0 = A \Rightarrow y = A\sqrt{x^2+1}$$

$$\frac{dy}{dx} - \frac{x}{x^2+1}y = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{x^2+1}y$$

$$\Rightarrow \frac{dy}{y} = \frac{x}{x^2+1}dx$$

$\therefore$ the equation is variable separable

$$\int\frac{dy}{y} = \int\frac{x}{x^2+1}dx \Rightarrow \ln|y| = \frac{1}{2}\ln(x^2+1) + c$$

$$\therefore y = A\sqrt{x^2+1}$$

c $\frac{1}{x}\frac{dy}{dx} + 2y = 3 \Rightarrow \frac{dy}{dx} + 2xy = 3x$

$\therefore$ the equation is linear

$$I(x) = e^{\int 2x dx} = e^{x^2}$$

Multiplying both sides of $\frac{dy}{dx} + 2xy = 3x$ by $I(x)$,

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = 3xe^{x^2} \Rightarrow \frac{d}{dx}\left(e^{x^2}y\right) = 3xe^{x^2}$$

which is an exact DE

$$e^{x^2}y = \frac{3}{2}e^{x^2} + c \Rightarrow y = \frac{3}{2} + Ae^{-x^2}$$

$$\frac{dy}{dx} + 2xy = 3x \Rightarrow \frac{dy}{dx} = x(3-2y)$$

$$\Rightarrow \frac{dy}{3-2y} = x\ dx$$

$\therefore$ the equation is variable separable

$$\int\frac{dy}{3-2y} = \int x\ dx \Rightarrow -\frac{1}{2}\ln|3-2y| = \frac{1}{2}x^2 + c$$

$$\int\frac{dy}{3-2y} = \int x\ dx \Rightarrow -\frac{1}{2}\ln|3-2y| = \frac{1}{2}x^2 + c$$

$$\Rightarrow \ln|3-2y| = -x^2 + c$$

$$\Rightarrow 3 - 2y = e^{-x^2+c}$$

$$\Rightarrow 2y = 3 - Ae^{-x^2}$$

$$\therefore y = \frac{3}{2} + Be^{-x^2}$$

5 $u = \frac{1}{y} \Rightarrow \frac{dy}{dx} = -\frac{1}{u^2}\frac{du}{dx}$

$$\frac{dy}{dx} + xy(1-y) = 0 \Rightarrow -\frac{1}{u^2}\frac{du}{dx} + x\frac{1}{u}\left(1 - \frac{1}{u}\right) = 0$$

$\Rightarrow \frac{du}{dx} - xu = -x$ which is a linear equation of the form $\frac{du}{dx} + p(x)u = q(u)$.

$$I(x) = e^{\int -x dx} = e^{-\frac{x^2}{2}}$$

Multiplying both sides of $\frac{du}{dx} - xu = -x$ by $I(x)$,

$$e^{-\frac{x^2}{2}}\frac{du}{dx} - xe^{-\frac{x^2}{2}}u = -xe^{-\frac{x^2}{2}} \Rightarrow \frac{d}{dx}\left(e^{-\frac{x^2}{2}}u\right) = -xe^{-\frac{x^2}{2}}$$

which is an exact DE

$$e^{-\frac{x^2}{2}}u = e^{-\frac{x^2}{2}} + A \Rightarrow u = 1 + Ae^{\frac{x^2}{2}}$$

$$\frac{1}{y} = 1 + Ae^{\frac{x^2}{2}} \Rightarrow y = \frac{1}{1 + Ae^{\frac{x^2}{2}}}$$

$$y(1) = 2 \Rightarrow 2 = \frac{1}{1 + Ae^{\frac{1}{2}}} \Rightarrow A = -\frac{1}{2}e^{-\frac{1}{2}}$$

$$y = \frac{1}{1 - \frac{1}{2}e^{\frac{x^2}{2} - \frac{1}{2}}}$$

Exercise 3F

1 Use cooling model equation $T(t) = ke^{\alpha t} + T_e$ and solve simultaneously the equations:

$$\begin{cases} T(0) = 24 \\ T(5) = 22 \\ T(10) = 20.4 \end{cases}$$

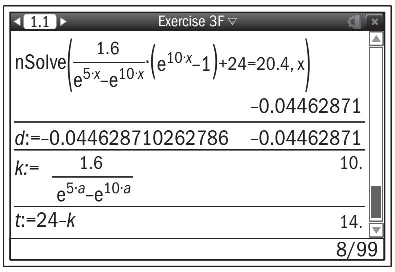
$$\Rightarrow \begin{cases} k + T_e = 24 \\ ke^{5\alpha} + T_e = 22 \\ ke^{10\alpha} + T_e = 20.4 \end{cases}$$

$$\Rightarrow \begin{cases} T_e = 24 - k \\ k\left(e^{5\alpha} - e^{10\alpha}\right) = 1.6 \\ ke^{10\alpha} + T_e = 20.4 \end{cases}$$

$$\Rightarrow \begin{cases} T_e = 24 - k \\ k = \frac{1.6}{e^{5\alpha} - e^{10\alpha}} \\ ke^{10\alpha} + 24 - k = 20.4 \end{cases}$$

$$\Rightarrow \begin{cases} T_e = 24 - k \\ k = \frac{1.6}{e^{5\alpha} - e^{10\alpha}} \\ \frac{1.6}{e^{5\alpha} - e^{10\alpha}}(e^{10\alpha} - 1) + 24 = 20.4 \end{cases}$$

to obtain $\alpha = -0.044628...$, $k = 10$ and $T_e = 14$.



So the temperature outside is approximately **14 degrees**

2 **a** $a = 5000,\ b = 0.04$

b $x(0) = 1000$

$$\frac{dx}{dt} = 5000 + 0.04x \Rightarrow \int \frac{dx}{5000 + 0.04x} = \int dt$$

$$\therefore \frac{1}{0.04} \ln|5000 + 0.04x| = t + c$$

$$\Rightarrow 5000 + 0.04x = Ae^{0.04t}$$

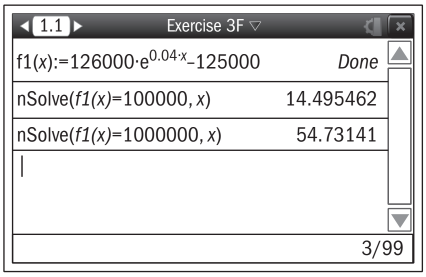
$$x(0) = 1000 \Rightarrow A = 5040$$

$$x = 126000e^{0.04t} - 125000$$

$$\text{or } x = 1000e^{0.04t} + 125000(e^{0.04t} - 1)$$

c $x(t) = 100000 \Rightarrow t = 14.5$ years.

$x(t) = 1000000 \Rightarrow t = 54.7$ year. So not in the first 50 years.

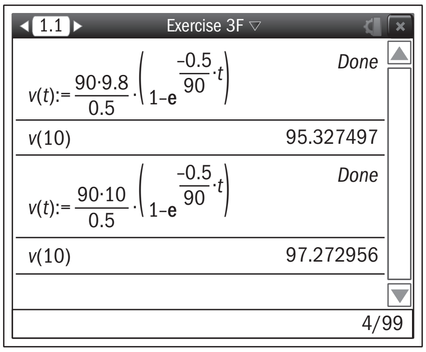


3 **a** Use $v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$ as deduced in example 16

with $k = 0.5$ and $m = 90$

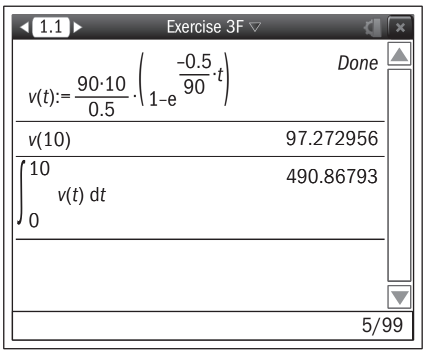
For g = 10, v = 97 ms^{-1}

Note: For $g = 9.8$, $v = 95.3\ ms^{-1}$



b $\int_0^1 \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) dt = 491$ **metres**

Note: when $g = 9.8$, s = 481 metres



c $\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) = \frac{mg}{k}$,

i.e., 90 ms^{-1} as $k = 10$

Note: when $g = 9.8$ $v = 88.2$

4 **a** Since solution is flowing out at 10l/m and the flow rate in is 5 l/m the quantity of water in the tank at any given instant will be 100-5t

$$\frac{dQ}{dt} = 10 \times 5 - \frac{Q}{100 - 5t} \times 10 \Rightarrow \frac{dQ}{dt} + \frac{2Q}{20 - t} = 50$$

$$I(t) = e^{\int \frac{2}{20-t} dt} = e^{-2\ln|20-t|} = \frac{1}{(20-t)^2}$$

$$\frac{1}{(20-t)^2}\frac{dQ}{dt} - \frac{2}{(20-t)^3}Q = \frac{50}{(20-t)^2}$$

$$\frac{d}{dt}\left(\frac{1}{(20-t)^2}Q\right) = \frac{50}{(20-t)^2}$$

$$\frac{1}{(20-t)^2}Q = \int \frac{50}{(20-t)^2} dt$$

$$\Rightarrow \frac{1}{(20-t)^2}Q = \frac{50}{20-t} + c$$

$$\therefore Q(t) = 50(20-t) + c(20-t)^2$$

$$Q(0) = 0 \Rightarrow c = -2.5$$

$$\therefore Q(t) = 50(20-t) - 2.5(20-t)^2$$

b $Q(10) = 250$ grams (3s.f.)

Exercise 3G

1 **a** $\frac{dy}{dx} = \frac{3x^2 - xy}{x^2} \Rightarrow \frac{dy}{dx} = \underbrace{3 - \frac{y}{x}}_{f\left(\frac{y}{x}\right)}$ $\therefore$ The DE is homogeneous.

b $2x\frac{dy}{dx} = x - y + 3 \Rightarrow \frac{dy}{dx} = \underbrace{\frac{1}{2} - \frac{y}{2x} + \frac{3}{2x}}_{\neq f\left(\frac{y}{x}\right)}$

$\therefore$ The DE is non-homogeneous

c $(x^2 + y)\frac{dy}{dx} = x \Rightarrow \frac{dy}{dx} = \frac{x}{x^2 + y} \Rightarrow \frac{dy}{dx} = \underbrace{\frac{\frac{1}{x}}{1 + \frac{y}{x}}}_{\neq f\left(\frac{y}{x}\right)}$

$\therefore$ The DE is non-homogeneous

2 **a** $\frac{dy}{dx} = \frac{3x^2 - xy}{x^2} \Rightarrow \frac{dy}{dx} = \underbrace{3 - \frac{y}{x}}_{f\left(\frac{y}{x}\right)}$

As $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$, substituting in the equation above,

$v + x\frac{dv}{dx} = 3 - v \Rightarrow x\frac{dv}{dx} = 3 - 2v$ which is variables separable DE

$$\int\frac{dv}{3-2v} = \int\frac{dx}{x} \Rightarrow -\frac{1}{2}\ln|3-2v| = \ln|x| + c$$

$$\therefore -\frac{1}{2}\ln\left|3 - \frac{2y}{x}\right| = \ln|x| + c \text{ or } y = \frac{x}{2}\left(3 + \frac{A}{x^2}\right)$$

3 a $2x\frac{dy}{dx} = x - y \Rightarrow \frac{dy}{dx} = \frac{x-y}{2x} \Rightarrow \frac{dy}{dx} = \underbrace{\frac{1}{2} - \frac{1}{2}\cdot\frac{y}{x}}_{f\left(\frac{y}{x}\right)}$

(homogeneous)

As $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$, substituting in the equation above,

Method 1:

$$v + x\frac{dv}{dx} = \frac{1}{2} - \frac{1}{2}v \Rightarrow x\frac{dv}{dx} = \frac{1}{2} - \frac{3}{2}v$$

(variable separable)

$$\int\frac{2}{1-3v}dv = \int\frac{dx}{x} \Rightarrow -\frac{2}{3}\ln|1-3v| = \ln|x| + c$$

$$\therefore (1-3v)^{-\frac{2}{3}} = Ax \Rightarrow \left(1 - \frac{3y}{x}\right)^{-\frac{2}{3}} = Ax$$

Method 2:

$$v + x\frac{dv}{dx} = \frac{1}{2} - \frac{1}{2}v \Rightarrow \frac{dv}{dx} + \frac{3}{2x}v = \frac{1}{2x}$$

(linear DE)

$$I(x) = e^{\int\frac{3}{2x}dx} = e^{\frac{3}{2}\ln(x)} = x^{\frac{3}{2}} \text{ or } \sqrt{x^3}$$

Multiplying both sides of $\frac{dv}{dx} + \frac{3}{2x}v = \frac{1}{2x}$ by $I(x)$,

$$x^{\frac{3}{2}}\frac{dv}{dx} + \frac{3}{2}x^{\frac{1}{2}}v = \frac{1}{2}x^{\frac{1}{2}} \Rightarrow \frac{d}{dx}\left(x^{\frac{3}{2}}v\right) = \frac{1}{2}x^{\frac{1}{2}}$$

$$\therefore x^{\frac{3}{2}}v = \frac{1}{3}x^{\frac{3}{2}} + c \Rightarrow v = \frac{1}{3} + cx^{-\frac{3}{2}} \Rightarrow y = \frac{1}{3}x + cx^{-\frac{1}{2}}$$

Use two different methods to solve the following differential equations

b $x\frac{dy}{dx} = 2x + 3y$

$$x\frac{dy}{dx} = 2x + 3y \Rightarrow \frac{dy}{dx} = \frac{2x+3y}{x} \Rightarrow \frac{dy}{dx} = \underbrace{2 + 3\cdot\frac{y}{x}}_{f\left(\frac{y}{x}\right)}$$

with final solution $y = Ax^3 - x$

Method 1:

$$v + x\frac{dv}{dx} = 2 + 3v$$

$$\Rightarrow x\frac{dv}{dx} = 2 + 2v$$

$$\Rightarrow \frac{1}{2}\int\frac{dv}{1+v} = \int\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\ln|1+v| = \ln|x| + c$$

$$\Rightarrow \ln|1+v| = \ln Ax^2$$

$$\Rightarrow 1 + \frac{y}{x} = Ax^2$$

$$\Rightarrow y = Ax^3 - x$$

Method 2:

$$v + x\frac{dv}{dx} = 2 + 3v$$

$$\Rightarrow x\frac{dv}{dx} = 2 + 2v$$

$$\Rightarrow \frac{dv}{dx} - \frac{2}{x}.v = \frac{2}{x}$$

$$I = e^{\int -\frac{2dx}{x}} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

$$\frac{1}{x^2}\frac{dv}{dx} - \frac{2}{x^3}.v = \frac{2}{x^3}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{v}{x^2}\right) = \frac{2}{x^3}$$

$$\Rightarrow \frac{v}{x^2} = \int\frac{2}{x^3}dx$$

$$\Rightarrow \frac{v}{x^2} = -\frac{1}{x^2} + c$$

$$\Rightarrow y = -x + cx^3$$

4 a $x^2\frac{dy}{dx} = 4x^2 + y^2 \Rightarrow \frac{dy}{dx} = \frac{4x^2+y^2}{x^2}$

$$\Rightarrow \frac{dy}{dx} = \underbrace{4 + \left(\frac{y}{x}\right)^2}_{f\left(\frac{y}{x}\right)}$$ **(homogeneous)**

As $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$, substituting in the equation above,

$$v + x\frac{dv}{dx} = 4 + v^2 \Rightarrow x\frac{dv}{dx} = 4 - v + v^2$$

(variable separable)

$$4 - v + v^2 = \left(v - \frac{1}{2}\right)^2 + \frac{15}{4}$$

$$\int\frac{1}{4-v-v^2}dv = \int\frac{dx}{x} \Rightarrow \int\frac{1}{\left(v-\frac{1}{2}\right)^2 + \frac{15}{4}}dv = \int\frac{dx}{x}$$

$$\frac{2}{\sqrt{15}}\arctan\left(\frac{2}{\sqrt{15}}\left(\frac{y}{x} - \frac{1}{2}\right)\right) = \ln|x| + c$$

b $(x^2+y^2)\frac{dy}{dx} = xy \Rightarrow \frac{dy}{dx} = \frac{xy}{x^2+y^2} \Rightarrow \frac{dy}{dx} = \underbrace{\frac{\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2}}_{f\left(\frac{y}{x}\right)}$

(homogeneous)

As $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$, substituting in the equation above,

$v + x\frac{dv}{dx} = \frac{v}{1+v^2} \Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^2} - v$

$\Rightarrow \int \frac{1+v^2}{v^3} dv = -\int \frac{dx}{x}$

$\Rightarrow -\frac{1}{2v^2} + \ln|v| = -\ln|x| + c$

$\Rightarrow \frac{1}{2v^2} = \ln|v| + \ln|x| + c$

$\Rightarrow \frac{1}{2v^2} = \ln A|vx|$

$\Rightarrow Avx = e^{\frac{1}{2v^2}}$

$\Rightarrow Ay = e^{\frac{x^2}{2y^2}}$

c $x(x^2+y^2)\frac{dy}{dx} = x^3 + 2x^2y + y^3$

$\Rightarrow \frac{dy}{dx} = \frac{x^3+2x^2y+y^3}{x^3+xy^2}$

$\frac{dy}{dx} = \underbrace{\frac{1+2\cdot\frac{y}{x}+\left(\frac{y}{x}\right)^3}{1+\left(\frac{y}{x}\right)^2}}_{f\left(\frac{y}{x}\right)}$ **(homogeneous)**

As $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$, substituting in the equation above,

$v + x\frac{dv}{dx} = \frac{1+2v+v^3}{1+v^2} \Rightarrow x\frac{dv}{dx} = \frac{1+v}{1+v^2}$

(variable separable)

$\int \frac{1+v^2}{1+v} dv = \int \frac{dx}{x} \Rightarrow \left(\int v - 1 + \frac{2}{1+v}\right) dv = \int \frac{dx}{x}$

$\Rightarrow \frac{1}{2}v^2 - v + 2\ln|v+1| = \ln|x| + c$

$\therefore (1+v)^2 e^{\frac{1}{2}v^2 - v} = Ax \Rightarrow \left(1+\frac{y}{x}\right)^2 e^{\frac{y^2-2xy}{2x^2}} = Ax$

5 a $(x^2-y^2)\frac{dy}{dx} = xy \Rightarrow \frac{dy}{dx} = \frac{xy}{x^2-y^2}$

$\frac{dy}{dx} = \frac{\frac{y}{x}}{1-\left(\frac{y}{x}\right)^2}$

As $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$, substituting in the equation above,

$v + x\frac{dv}{dx} = \frac{v}{1-v^2} \Rightarrow x\frac{dv}{dx} = \frac{v^3}{1-v^2}$

(variable separable)

$\int \frac{1-v^2}{v^3} dv = \int \frac{dx}{x} \Rightarrow \left(\int \frac{1}{v^3} - \frac{1}{v}\right) dv = \int \frac{dx}{x}$

$\Rightarrow -\frac{1}{2v^2} - \ln|v| = \ln|x| + c$

$v(4) = \frac{1}{2} \Rightarrow -2 - \ln\frac{1}{2} = \ln 4 + c \Rightarrow c = -2 - \ln 2$

$\therefore -\frac{1}{2v^2} - \ln|v| = \ln|x| - \frac{1}{8} - \ln 2$

$\therefore -\frac{1}{2}\left(\frac{x}{y}\right)^2 - \ln\left|\frac{y}{x}\right| - \ln|x| + 2 + \ln 2 = 0$ **or**

$-\frac{1}{2}\left(\frac{x}{y}\right)^2 - \ln|y| + 2 + \ln 2 = 0$

b $x^2\frac{dy}{dx} = x^2 - xy + y^2 \Rightarrow \frac{dy}{dx} = \frac{x^2 - xy + y^2}{x^2}$

As $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$, substituting in the equation above,

$v + x\frac{dv}{dx} = 1 - v + v^2 \Rightarrow x\frac{dv}{dx} = 1 - 2v + v^2$

(variable separable)

$\int \frac{1}{(1-v)^2} dv = \int \frac{dx}{x} \Rightarrow -\frac{1}{1-v} = \ln|x| + c$

$v(1) = 0 \Rightarrow c = -1$

$\frac{1}{v-1} = \ln|x| - 1 \Rightarrow v = \frac{\ln|x|}{\ln|x|-1}$

$\therefore y = \frac{x\ln|x|}{\ln|x| - 1}$

c $x^2\frac{dy}{dx} = x^2 + xy + y^2 \Rightarrow \frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$

As $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$, substituting in the equation above,

$v + x\frac{dv}{dx} = 1 + v + v^2 \Rightarrow x\frac{dv}{dx} = 1 + v^2$

(variable separable)

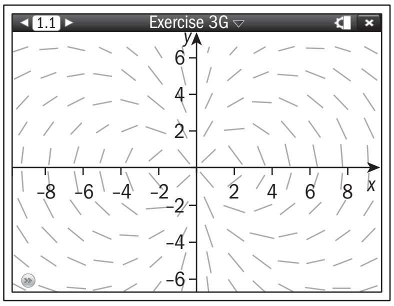
$\int \frac{1}{1+v^2} dv = \int \frac{dx}{x} \Rightarrow \arctan(v) = \ln|x| + c$

$v(1) = 0 \Rightarrow c = 0$

$\therefore \arctan(v) = \ln|x|$

$\therefore \arctan\left(\frac{y}{x}\right) = \ln|x|$

6 a



b $\dfrac{dy}{dx} = \dfrac{2y^2 - x^2}{3xy}$

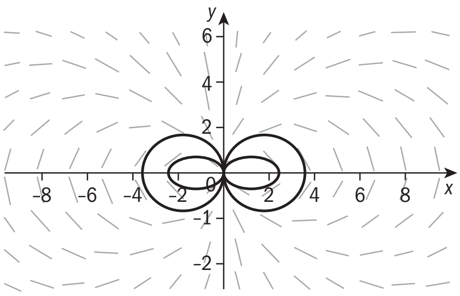
As $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, substituting in the equation above,

$v + x\dfrac{dv}{dx} = \dfrac{2v^2 - 1}{3v} \Rightarrow x\dfrac{dv}{dx} = \dfrac{-v^2 - 1}{3v}$

(variable separable)

$\int \dfrac{3v}{v^2 + 1}dv = -\int \dfrac{dx}{x} \Rightarrow -\dfrac{3}{2}\ln(v^2 + 1) = -\ln|x| + c$

$\therefore \left(\dfrac{y}{x}\right)^2 + 1 = \dfrac{A}{\sqrt[3]{x^2}}$



7 (extension)

a
- terms have different degrees ∴ non-homogeneous DE
- the equation contains term in y^2 ∴ non-linear DE
- it is not possible to re-arrange the equation to the form $\dfrac{dy}{dx} = \dfrac{f(y)}{g(x)}$ ∴ non-separable DE

b $v = y^2 \Rightarrow y = v^{\frac{1}{2}}$ and $\dfrac{dy}{dx} = \dfrac{dy}{dv}\cdot\dfrac{dv}{dx} = \dfrac{1}{2}v^{-\frac{1}{2}}\dfrac{dv}{dx}$

$\therefore 2xy\dfrac{dy}{dx} = 1 + y^2 - x^2 \Rightarrow x\underbrace{v^{\frac{1}{2}}v^{-\frac{1}{2}}}_{1}\dfrac{dv}{dx} = 1 + v - x^2$

$\therefore x\dfrac{dv}{dx} = 1 + v - x^2$ QED

c $u = x^2 \Rightarrow x = u^{\frac{1}{2}}$ and $\dfrac{dv}{dx} = \dfrac{dv}{du}\dfrac{du}{dx} \Rightarrow \dfrac{dv}{dx} = 2x\dfrac{dv}{du}$

$\therefore x\dfrac{dv}{dx} = 1 + v - x^2 \Rightarrow 2u\dfrac{dv}{du} = 1 + v - u$

$\therefore 2u\dfrac{dv}{du} = 1 + v - u$ QED

d $w = 1 + v \Rightarrow v = 1 - w$ and

$\dfrac{dw}{du} = \underbrace{\dfrac{dw}{dv}}_{1}\dfrac{dv}{du} \Rightarrow \dfrac{dw}{du} = \dfrac{dv}{du}$

$\therefore 2u\dfrac{dv}{du} = 1 + v - u \Rightarrow 2u\dfrac{dw}{du} = w - u$ QED

e $2u\dfrac{dw}{du} = w - u \Rightarrow \dfrac{dw}{du} = \underbrace{\dfrac{1}{2}\cdot\dfrac{w}{u} - 1}_{f\left(\frac{w}{u}\right)}$ **(homogeneous)**

$2u\dfrac{dw}{du} = w - u \Rightarrow \dfrac{dw}{du}\underbrace{-\dfrac{1}{2u}}_{p(u)}w = \underbrace{\dfrac{1}{2}}_{q(u)}$ **(linear)**

f $2u\dfrac{dw}{du} = w - u \Rightarrow \dfrac{dw}{du} = \dfrac{w - u}{2u} \Rightarrow \dfrac{dw}{du} = \dfrac{1}{2}\left(\dfrac{w}{u} - 1\right)$

As $w = ut \Rightarrow \dfrac{dw}{du} = t + u\dfrac{dt}{du}$, substituting in the equation above,

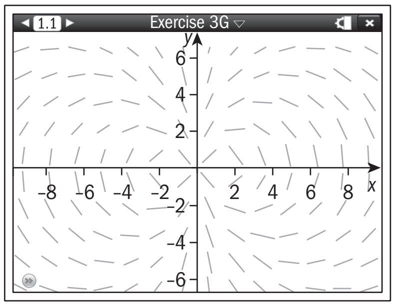
$t + u\dfrac{dt}{du} = \dfrac{t - 1}{2} \Rightarrow u\dfrac{dt}{du} = -\dfrac{t + 1}{2}$

(variable separable)

$\int \dfrac{dt}{t + 1} = -\dfrac{1}{2}\int \dfrac{1}{u}du \Rightarrow \ln|t + 1| = -\dfrac{1}{2}\ln|x| + c$

$t + 1 = \dfrac{A}{\sqrt{u}} \Rightarrow \dfrac{w}{u} + 1 = \dfrac{A}{\sqrt{u}} \Rightarrow \dfrac{v + 1}{u} + 1 = \dfrac{A}{\sqrt{u}}$

$\therefore \dfrac{y^2 + 1}{x^2} + 1 = \dfrac{A}{x} \Rightarrow x^2 + y^2 - Ax + 1 = 0$ circles



8 (extension)

a $x = u - 1;\ y = v + 1 \Rightarrow \dfrac{dv}{du} = \dfrac{dy}{dx}$

$\therefore \dfrac{dy}{dx} = \dfrac{x - y + 2}{x + y} \Rightarrow \dfrac{dv}{du} = \dfrac{u - v}{u + v} \Rightarrow \dfrac{dv}{du} = \dfrac{1 - \frac{v}{u}}{1 + \frac{v}{u}}$

(homogeneous)

b As $v = ut \Rightarrow \dfrac{dv}{du} = t + u\dfrac{dt}{du}$, substituting in the equation above,

$t + u\dfrac{dt}{du} = \dfrac{1 - t}{1 + t} \Rightarrow u\dfrac{dt}{du} = \dfrac{1 - 2t - t^2}{1 + t}$

(variable separable)

$$\int \frac{2t+2}{t^2+2t-1}\,dt = -2\int \frac{1}{u}\,du$$

$$\Rightarrow \ln|t^2+2t-1| = -2\ln|u| + c$$

$$\therefore t^2 + 2t - 1 = \frac{A}{u^2} \Rightarrow \left(\frac{y-1}{x+1}\right)^2 + 2\left(\frac{y-1}{x+1}\right) - 1$$

$$= \frac{A}{(x+1)^2}$$

$$\therefore (y-1)^2 + 2(y-1)(x+1) - (x+1)^2 = A$$

9 (extension)

a $v = \frac{1}{y};\ \frac{dy}{dx} + 2y = y^2 \Rightarrow -\frac{1}{v^2}\frac{dv}{dx} + \frac{2}{v} = \frac{1}{v^2}$

$\therefore \frac{dv}{dx} - 2v = -1$ **(linear DE)**

$I(x) = e^{\int -2dx} = e^{-2x}$

Multiplying both sides of $\frac{dv}{dx} - 2v = -1$ by $I(x)$

$e^{-2x}\frac{dv}{dx} - 2e^{-2x}v = -e^{-2x} \Rightarrow \frac{d}{dx}(e^{-2x}v) = -e^{-2x}$

$e^{-2x}v = -\frac{1}{2}e^{-2x} + c \Rightarrow v = -\frac{1}{2} + c \cdot e^{2x}$

$\therefore y = \frac{2}{1 + Ae^{2x}}$

b $v = \frac{1}{y^2};\ \frac{dy}{dx} - y = y^3 \Rightarrow -v^{-\frac{3}{2}}\frac{dv}{dx} - v^{-\frac{1}{2}} = v^{-\frac{3}{2}}$

$\therefore \frac{dv}{dx} + 2v = -2$ **(linear DE)**

$I(x) = e^{\int 2dx} = e^{2x}$

Multiplying both sides of $\frac{dv}{dx} + 2v = -2$ by $I(x)$

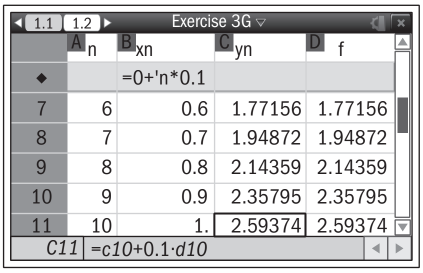
$e^{2x}\frac{dv}{dx} + 2e^{2x}v = -2e^{2x} \Rightarrow \frac{d}{dx}(e^{2x}v) = -2e^{2x}$

$e^{2x}v = -e^{2x} + A \Rightarrow v = -1 + Ae^{-2x}$

$\therefore y^2 = \frac{1}{Ae^{-2x} - 1}$

Investigation – Euler's Number

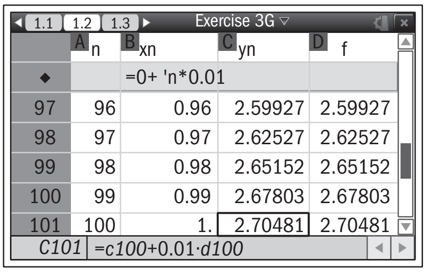
e.g. approximations for $h = 0.1$, 0.01 and 0.001; the approximation of e gets better



Exercise 3G

	A n	B xn	C yn	D f
◆		=0+'n*0.1		
7	6	0.6	1.77156	1.77156
8	7	0.7	1.94872	1.94872
9	8	0.8	2.14359	2.14359
10	9	0.9	2.35795	2.35795
11	10	1.	2.59374	2.59374

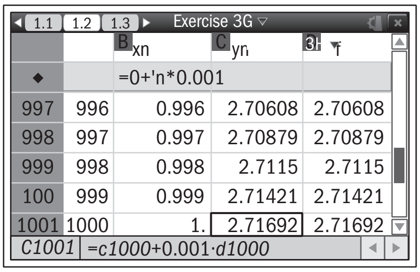
C11 =c10+0.1·d10



Exercise 3G

	A n	B xn	C yn	D f
◆		=0+ 'n*0.01		
97	96	0.96	2.59927	2.59927
98	97	0.97	2.62527	2.62527
99	98	0.98	2.65152	2.65152
100	99	0.99	2.67803	2.67803
101	100	1.	2.70481	2.70481

C101 =c100+0.01·d100



Exercise 3G

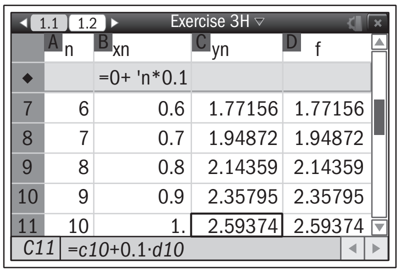
		B xn	C yn	D f
◆		=0+'n*0.001		
997	996	0.996	2.70608	2.70608
998	997	0.997	2.70879	2.70879
999	998	0.998	2.7115	2.7115
100	999	0.999	2.71421	2.71421
1001	1000	1.	2.71692	2.71692

C1001 =c1000+0.001·d1000

and for $h = 0.0001$, $y(1) = 2.7105461\ldots$ and this is a worse approx. of e

Exercise 3H

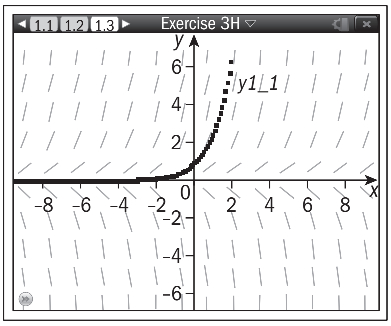
1



Exercise 3H

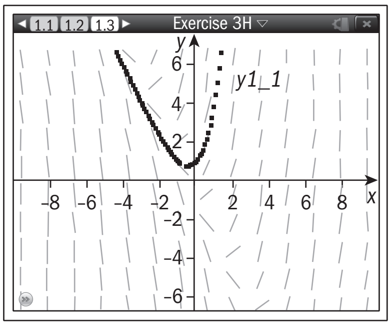
	A n	B xn	C yn	D f
◆		=0+ 'n*0.1		
7	6	0.6	1.77156	1.77156
8	7	0.7	1.94872	1.94872
9	8	0.8	2.14359	2.14359
10	9	0.9	2.35795	2.35795
11	10	1.	2.59374	2.59374

C11 =c10+0.1·d10



$y(1) = 2.59$

2 a, b and d



c

	A n	B xn	C yn	D f
◆		=0+ 'n*0.0		
997	996	0.996	4.12625	6.11825
998	997	0.997	4.13237	6.12637
999	998	0.998	4.13849	6.13449
100	999	0.999	4.14463	6.14263
1001	1000	1.	4.15077	6.15077

C1001 =c1000+0.001·d1000

$y(1) = 4.1507...$ when $h = 0.001$

e $\frac{dy}{dx} = 2x + y \Rightarrow \frac{dy}{dx} - y = 2x$ **(linear DE)**

$I(x) = e^{\int -1dx} = e^{-x}$

Multiplying both sides of $\frac{dy}{dx} - y = 2x$ by $I(x)$,

$e^{-x}\frac{dy}{dx} - e^{-x}y = 2xe^{-x} \Rightarrow \frac{d}{dx}(e^{-x}y) = 2xe^{-x}$

which is an exact DE

$\therefore e^{-x}y = 2\int xe^{-x}dx \Rightarrow e^{-x}y = -2(x+1)e^{-x} + c$

$\therefore y = -2(x+1) + c\cdot e^{x}$

$y(0) = 1 \Rightarrow c = 2$

$\therefore y = -2(x+1) + 2e^{x}$ **and** $y(1) = -4 + 2e$

f The approximation gives the correct answer to 3 s.f.

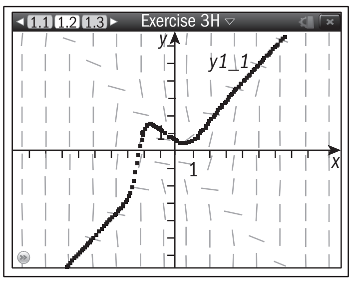
3 a

	A x	B y	C f	D
◆				
2	-1.8	1.6	0.68	
3	-1.6	1.736	-0.453696	
4	-1.4	1.64526...	-0.74688...	
5	-1.2	1.49588...	-0.79766...	
6	-1.	1.33635...	-0.78583...	

B6 =b5+0.2·c5

$y(-1) = 1.34$

b



4 a

	A xn	B yn	C f	D
◆				
7	2.6	1.16285...	0.21037...	
8	2.7	1.18389...	0.19593...	
9	2.8	1.20349...	0.18243...	
10	2.9	1.22173...	0.16980...	
11	3.	1.23871...	0.15799...	

B11 =b10+0.1·c10

$y = 1.24$

b $\frac{dy}{dx} = \frac{x}{\sin y \times e^{x}}$

$\Rightarrow \int \sin y dy = \int xe^{-x}dx$

$\Rightarrow \cos y = e^{-x}(x+1) + c$ (integrating RHS by parts)

$y(2) = 1 \Rightarrow c = \cos 1 - 3e^{-2}$

$\therefore y = \arccos(e^{-x}(x+1) + \cos 1 - 3e^{-2})$

$\Rightarrow y(3) = 1.2308...$

$\therefore$ exact value is $y = 1.23$

c e.g. Graphical method not reliable to predict values which are not close to initial value.

5 a

	A xn	B yn	C f	D
◆				
7	2.6	0.62902...	-0.42151...	
8	2.7	0.58687...	-0.38228...	
9	2.8	0.54864...	-0.34755...	
10	2.9	0.51389...	-0.31674...	
11	3.	0.48221...	-0.28932...	

B11 =b10+0.1·c10

$y = 0.48$

b $\frac{dy}{dx} = \frac{-2xy}{1+x^2}$ **(separable)**

$\Rightarrow \int \frac{dy}{y} = -\int \frac{2x}{1+x^2}dx$

$\therefore \ln|y| = -\ln(1+x^2) + c \Rightarrow y = \frac{A}{1+x^2}$

$y(2) = 1 \Rightarrow A = 5$

$\therefore f(x) = \frac{5}{1+x^2}$

c $f(3) = \frac{5}{1+3^2} = 0.5 > 0.48.$

So the approximation gives the correct value to 1 d.p.

6 a

	A xn	B yn	C f	D
◆				
1	0.	1.	1.	
2	0.25	1.25	1.5625	
3	0.5	1.640625	2.69165...	
4	0.75	2.31353...	5.35245...	
5	...	...	...	

B4 =b3+0.25·c3

b $\frac{dy}{dx} = y^2 \Rightarrow \int \frac{dy}{y^2} = \int dx \Rightarrow -\frac{1}{y} = x + c$

$\therefore y = -\frac{1}{x+c}$

$y(0) = 1 \Rightarrow c = -1$

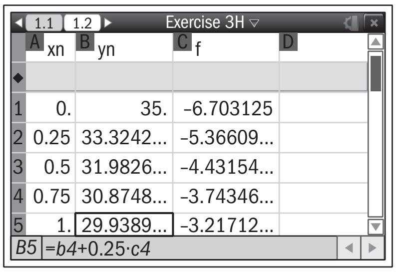
$\therefore y = \frac{1}{1-x}$

$y(0.25) = 1.(3),\ y(0.5) = 2$ **and** $y(0.75) = 4$

c The error increased as the value of x increased.

7 a The rate of decrease of temperature of a body is proportional to the fourth power of the temperature of the body.

b



1.1 1.2 Exercise 3H

	A xn	B yn	C f	D
◆				
1	0.	35.	-6.703125	
2	0.25	33.3242...	-5.36609...	
3	0.5	31.9826...	-4.43154...	
4	0.75	30.8748...	-3.74346...	
5	1.	29.9389...	-3.21712...	

B5 =b4+0.25·c4

T(1) = 29.9

c $\ln\frac{T+M}{T-M} + 2\arctan\frac{T}{M} - 4M^3kt = c$

$$\left(\frac{1}{T+M} - \frac{1}{T-M} + \frac{\frac{2}{M}}{1+\left(\frac{T}{M}\right)^2}\right)\frac{dT}{dt} - 4M^3k = 0$$

$$\Rightarrow \left(\frac{-2M}{T^2-M^2} + \frac{2M}{M^2+T^2}\right)\frac{dT}{dt} - 4M^3k = 0$$

$$\Rightarrow \left(\frac{-1}{T^4-M^4}\right)\frac{dT}{dt} - k = 0$$

$$\therefore \frac{dT}{dt} = -4k(T^4 - M^4) \text{ QED}$$

$$\ln\left(\frac{T+20}{T-20}\right) + 2\arctan\left(\frac{T}{20}\right) - \frac{4}{25}t = c$$

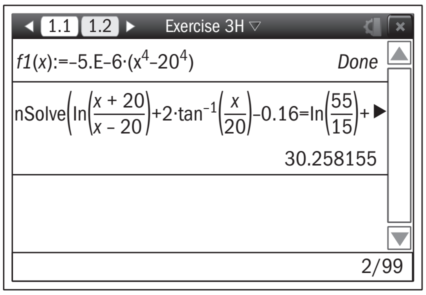
$$T(0) = 35 \Rightarrow c = \ln\left(\frac{11}{3}\right) + 2\arctan\left(\frac{7}{4}\right)$$

$$\ln\left(\frac{T+20}{T-20}\right) + 2\arctan\left(\frac{T}{20}\right) - \frac{4}{25}t$$

$$= \ln\left(\frac{11}{3}\right) + 2\arctan\left(\frac{7}{4}\right)$$

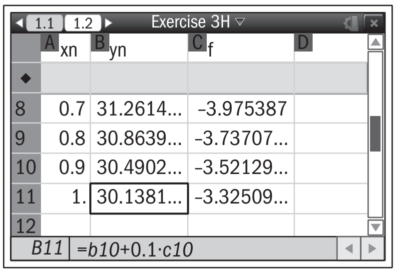
$$t = 1 \Rightarrow \ln\left(\frac{T+20}{T-20}\right) + 2\arctan\left(\frac{T}{20}\right) - \frac{4}{25}$$

$$= \ln\left(\frac{11}{3}\right) + 2\arctan\left(\frac{7}{4}\right)$$



T(1) = 30.3

d decrease the step (e.g. for h = 0.1, T(1) = 30.1)

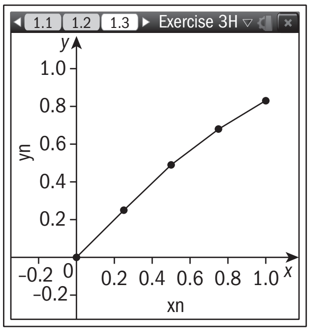


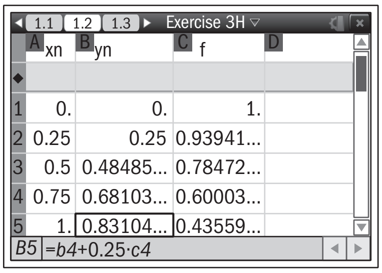
1.1 1.2 Exercise 3H

	A xn	B yn	C f	D
◆				
8	0.7	31.2614...	-3.975387	
9	0.8	30.8639...	-3.73707...	
10	0.9	30.4902...	-3.52129...	
11	1.	30.1381...	-3.32509...	
12				

B11 =b10+0.1·c10

8 a



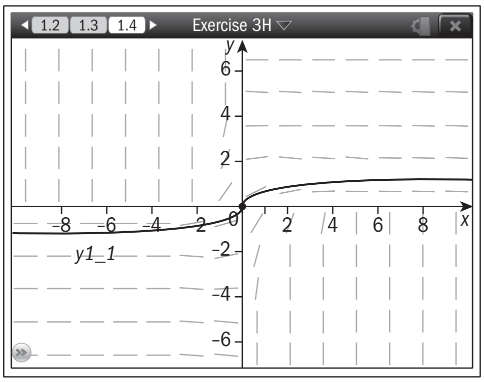


1.1 1.2 1.3 Exercise 3H

	A xn	B yn	C f	D
◆				
1	0.	0.	1.	
2	0.25	0.25	0.93941...	
3	0.5	0.48485...	0.78472...	
4	0.75	0.68103...	0.60003...	
5	1.	0.83104...	0.43559...	

B5 =b4+0.25·c4

b



Review exercise

1 a $\frac{dT}{dt} = k(T-23) \Rightarrow \int \frac{dT}{T-23} = k\int dt$

$\ln|T-23| = kt + c \Rightarrow |T-23| = e^{kt+c}$

$\therefore T = Ae^{kt} + 23$

b i $T(0) = 90 \Rightarrow A + 23 = 90 \Rightarrow A = 67$

$T(10) = 70 \Rightarrow 67e^{10k} + 23 = 70$

$\Rightarrow k = \frac{1}{10}\ln\left(\frac{47}{67}\right)$

ii $T(t) = 40 \Rightarrow t = 38.7$

2 $\frac{dA}{dt} = -kA \Rightarrow \int \frac{dA}{A} = -k\int dt$

$\ln|A| = -kt + c \Rightarrow A = e^{-kt+c}$

$A(0) = A_0 \Rightarrow e^c = A_0$

$\therefore A = A_0 e^{-kt}$

$A = \frac{1}{2}A_0 \Rightarrow A_0 e^{-kt} = \frac{1}{2}A_0 \Rightarrow kt = \ln 2 \Rightarrow t = \frac{\ln 2}{k}$

$A_0 = 50$ and $A(10) = 47 \Rightarrow k = -\frac{1}{10}\ln\left(\frac{47}{50}\right)$

$\therefore t = 112.02\ldots = 112$ to 3 s.f.

3 $x\frac{dy}{dx} - y^2 = 1 \Rightarrow \frac{dy}{dx} = \frac{y^2+1}{x} \Rightarrow \int \frac{dy}{y^2+1} = \int \frac{dx}{x}$

$\therefore \arctan(y) = \ln|x| + c$

$y(3) = 0 \Rightarrow \arctan(0) = \ln(3) + c \Rightarrow c = -\ln(3)$

$\therefore \arctan(y) = \ln|x| - \ln(3) \Rightarrow y = \tan\left(\ln\left|\frac{x}{3}\right|\right)$

4 a $2x^2\frac{dy}{dx} = x^2 + y^2 \Rightarrow \frac{dy}{dx} = \underbrace{\frac{x^2+y^2}{2x^2}}_{f(x,y)}$

(Note that $f(\lambda x, \lambda y) = f(x,y)$)

As $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$, substituting in the equation above,

$v + x\frac{dv}{dx} = \frac{1+v^2}{2}$ QED

b $v + x\frac{dv}{dx} = \frac{1+v^2}{2} \Rightarrow \frac{dv}{dx} = \frac{1}{x}\left(\frac{1+v^2}{2} - v\right)$

$\Rightarrow \frac{dv}{dx} = \frac{1}{x}\frac{(v-1)^2}{2} \Rightarrow \frac{2}{(v-1)^2}dv = \frac{dx}{x}$

$\therefore$ the variables can be separated QED

c $\int \frac{2}{(v-1)^2}dv = \int \frac{dx}{x} \Rightarrow -\frac{2}{v-1} = \ln|x| + c$

$\Rightarrow v = -\frac{2}{\ln|x| + c} - 1$

$\therefore y = -x\left(\frac{2}{\ln|x| + c} - 1\right)$

5 $I(x) = e^{\int -2\tan(x)dx} = \cos^2(x)$

Multiplying both sides of $\frac{dy}{dx} - 2\tan(x)\cdot y = 1$ by $I(x)$,

$\cos^2(x)\cdot\frac{dy}{dx} - 2\cos(x)\sin(x)\cdot y = \cos^2(x)$ which is an exact DE

$\therefore \cos^2(x)\cdot y = \int \cos^2(x)dx$

As $\cos^2(x) = \frac{\cos(2x)+1}{2}$ it follows that

$\cos^2(x)\cdot y = \int \frac{\cos(2x)+1}{2}dx$

$\Rightarrow \cos^2(x)\cdot y = \frac{1}{4}\sin(2x) + \frac{1}{2}x + c$

$y\left(\frac{\pi}{4}\right) = 2 \Rightarrow \frac{1}{4} + \frac{\pi}{8} + c = 2 \Rightarrow c = \frac{3}{4} - \frac{\pi}{8}$

$y = \frac{1}{2}\tan(x) + \left(\frac{1}{2}x + \frac{3}{4} - \frac{\pi}{8}\right)\sec^2(x)$

6 a $\frac{dy}{dx} = \underbrace{\frac{3y^2+x^2}{2xy}}_{f(x,y)}$ (Note that $f(\lambda x, \lambda y) = f(x,y)$)

As $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$, substituting in the equation above,

$v + x\frac{dv}{dx} = \frac{3v^2+1}{2v} \Rightarrow x\frac{dv}{dx} = \frac{v^2+1}{2v}$ which is variables separable DE

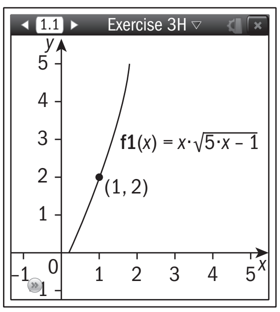
b $\int \frac{2v}{v^2+1}dv = \int \frac{1}{x}dx \Rightarrow \ln(v^2+1) = \ln|x| + c$

$\therefore v^2 + 1 = A|x| \Rightarrow \left(\frac{y}{x}\right)^2 + 1 = A|x|$

c $y(1) = 2 \Rightarrow A = 5$

$\left(\frac{y}{x}\right)^2 + 1 = 5x \Rightarrow y = x\sqrt{5x-1},\ x \geq \frac{1}{5}$

d



7 $(x^2 + y^2) + 2xy\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$

(homogeneous DE)

As $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, substituting in the equation above,

$v + x\dfrac{dv}{dx} = v + \dfrac{v^2+1}{2v} \Rightarrow x\dfrac{dv}{dx} = -\dfrac{3v^2+1}{2v}$ which is

variables separable DE

$\int \dfrac{2v}{3v^2+1}\,dv = -\int \dfrac{1}{x}\,dx \Rightarrow \dfrac{1}{3}\ln(3v^2+1) = -\ln|x| + c$

$x > 0 \Rightarrow 3v^2 + 1 = \dfrac{a}{x^3} \Rightarrow 3\left(\dfrac{y}{x}\right)^2 + 1 = \dfrac{a}{x^3},\ a = e^c$

$\therefore x^3 + 3xy^2 = a,\ a > 0$ QED

8 a $\dfrac{dy}{dx} = y\tan(x) + 1 \Rightarrow \cos(x)\cdot\dfrac{dy}{dx}$

$= \cos(x)\cdot(y\tan(x) + 1)$

$\Rightarrow \cos(x)\dfrac{dy}{dx} - \sin(x)\cdot y = \cos(x)$ (exact DE)

$\therefore I(x) = \cos(x)$ is an integrating factor.

b $\cos(x)\dfrac{dy}{dx} - \sin(x)\cdot y = \cos(x)$

$\Rightarrow \cos(x)\cdot y = \int \cos(x)\,dx$

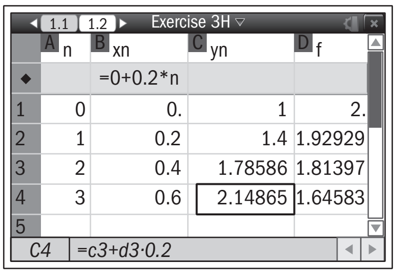
$\cos(x)\cdot y = \sin(x) + c$

$\therefore y = \tan(x) + \dfrac{c}{\cos(x)}$

$y(0) = 2 \Rightarrow c = 2$

$\therefore y = \tan(x) + \dfrac{2}{\cos(x)}$ **(or** $\therefore y = \tan(x) + 2\sec(x)$**)**

9 a



n	xn	yn	f
	=0+0.2*n		
0	0.	1	2.
1	0.2	1.4	1.92929
2	0.4	1.78586	1.81397
3	0.6	2.14865	1.64583

C4 =c3+d3·0.2

Answer: 2.149

b $\dfrac{dy}{dx} + \dfrac{xy}{4-x^2} = 2$ is a linear DE

$I(x) = e^{\int \frac{x}{4-x^2}dx} = e^{-\frac{1}{2}\ln|4-x^2|} = \dfrac{1}{\sqrt{4-x^2}}$

Multiplying both sides of $\dfrac{dy}{dx} + \dfrac{xy}{4-x^2} = 2$ by $I(x)$,

$\dfrac{1}{\sqrt{4-x^2}}\left(\dfrac{dy}{dx} + \dfrac{xy}{4-x^2}\right) = \dfrac{2}{\sqrt{4-x^2}}$

$\Rightarrow \dfrac{d}{dx}\left(\dfrac{1}{\sqrt{4-x^2}}\cdot y\right) = \dfrac{2}{\sqrt{4-x^2}}$

which is an exact DE

$\therefore \dfrac{1}{\sqrt{4-x^2}}\cdot y = \int \dfrac{2}{\sqrt{4-x^2}}\,dx$

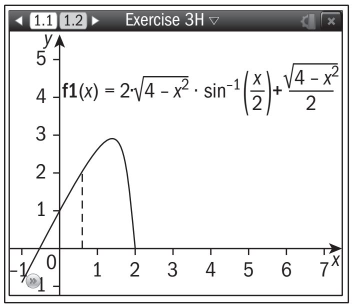
$\Rightarrow \dfrac{1}{\sqrt{4-x^2}}\cdot y = 2\arcsin\left(\dfrac{x}{2}\right) + c$

$y(0) = 1 \Rightarrow c = \dfrac{1}{2}$

$\therefore y = 2\sqrt{4-x^2}\,\arcsin\left(\dfrac{x}{2}\right) + \dfrac{1}{2}\sqrt{4-x^2}$

c $y(0) = 2.117$ (to 3 d.p.) (use GDC)

d



The graph is increasing for $0 < x < 0.6$ which explains why the approximate solution is greater than the exact one.

10 a $\dfrac{dV}{dt} = -\alpha V \Rightarrow \int \dfrac{dV}{V} = -\alpha\int dt$

$\Rightarrow \ln V = -\alpha T + c$

$\therefore V = Ae^{-\alpha t}$

$V(0) = V_0 \Rightarrow A = V_0$

$\therefore V = V_0e^{-\alpha t}$

b $V = \dfrac{V_0}{3} \Rightarrow V_0e^{-\alpha t} = \dfrac{V_0}{3} \Rightarrow -\alpha t = \ln\left(\dfrac{1}{3}\right)$

$\therefore t = \dfrac{\ln 3}{\alpha}$

11 $xy\dfrac{dy}{dx} = 1 + y^2$ is a variable separable DE

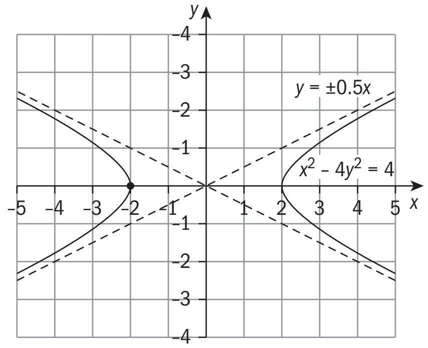
$\int \dfrac{y}{1+y^2}\,dy = \int \dfrac{1}{x}\,dx \Rightarrow \dfrac{1}{2}\ln(y^2+1) = \ln|x| + c$

$y(2) = 0 \Rightarrow c = \ln\left(\dfrac{1}{2}\right)$

$\frac{1}{2}\ln(y^2+1) = \ln|x| + \ln\left(\frac{1}{2}\right) \Rightarrow \sqrt{y^2+1} = \frac{x}{2}$

$\therefore x^2 - 4y^2 = 4$ or

$y = \pm\frac{\sqrt{x^2-4}}{2}, D =]-\infty, -2[\cup]2, +\infty[$



12 $kx = mv\frac{dv}{dx} \Rightarrow k\int x\,dx = m\int v\,dv$

$k\frac{x^2}{2} = m\frac{v^2}{2} + c$

$v(0) = v_0 \Rightarrow m\frac{v_0^2}{2} + c = 0 \Rightarrow c = -m\frac{v_0^2}{2}$

$k\frac{x^2}{2} = m\frac{v^2}{2} - m\frac{v_0^2}{2} \Rightarrow kx^2 = mv^2 - mv_0^2$

$v = \pm\sqrt{\frac{kx^2 + mv_0^2}{m}}$

$\therefore v(1) = \pm\sqrt{\frac{k + mv_0^2}{m}}$

13 $\frac{1}{4x-x^2} = \frac{a}{x} + \frac{b}{x-4} \Rightarrow \frac{1}{4x-x^2} = \frac{(a+b)x-4a}{4x-x^2}$

$\Rightarrow a = -b$ and $4b = 1$

$\therefore a = -\frac{1}{4}, b = \frac{1}{4}$

$\frac{dx}{dt} = kx(4-x) \Rightarrow \frac{dx}{4x-x^2} = k\,dt$

(variable separable DE)

$\Rightarrow \frac{1}{4}\int\left(-\frac{1}{x} - \frac{1}{4-x}\right)dx = k\int dt$

$-\frac{1}{4}(\ln x + \ln(4-x)) = kt + c$

$\therefore 4x - x^2 = Ae^{-4kt}, 0 < x < 4$

14 a $\frac{d^2y}{dx^2} = \frac{1}{125l^3}(x^2 - lx)$

$z = \frac{1}{125l^3}\left(\frac{x^3}{3} - \frac{lx^2}{2}\right) + \frac{1}{1500}$

$\Rightarrow \frac{dz}{dx} = \frac{1}{125l^3}\left(\frac{3\cdot x^2}{3} - \frac{2lx}{2}\right)$

$\therefore \frac{dz}{dx} = \frac{1}{125l^3}(x^2 - lx)$ QED

b $\frac{dy}{dx} = z \Rightarrow \frac{dy}{dx} = \frac{1}{125l^3}\left(\frac{x^3}{3} - \frac{lx^2}{2}\right) + \frac{1}{1500}$

$y = \frac{1}{125l^3}\left(\frac{x^4}{12} - \frac{lx^3}{6}\right) + \frac{x}{1500} + c$

$y(0) = 0 \Rightarrow c = 0$

$\therefore y = \frac{1}{125l^3}\left(\frac{x^4}{12} - \frac{lx^3}{6}\right) + \frac{x}{1500}$ or

$\therefore y = \frac{1}{125l^3}\left(\frac{x^4}{12} - \frac{lx^3}{6} + \frac{l^3x}{12}\right)$

c $\frac{dy}{dx} = \frac{1}{125l^3}\left(\frac{x^3}{3} - \frac{lx^2}{2} + \frac{l^3}{12}\right) \Rightarrow \frac{d^2y}{dx^2}$

$= \frac{1}{125l^3}(x^2 - lx)$

d $y(l) = \frac{1}{125l^3}\left(\frac{l^4}{12} - \frac{l^4}{6} + \frac{l^4}{12}\right) = 0$

e $y(2) = \frac{1}{125\times 4^3}\left(\frac{2^4}{12} - \frac{4\times 2^3}{6} + \frac{4^3\times 2}{12}\right) = \frac{1}{1200}$;

maximum distance below [AB]

4 The finite in the infinite

Exercise 4A

1 a $\lim_{n\to\infty} 3^n \neq 0$, hence the series diverges.

b $\sum_{n\to 0}^{\infty} \frac{1}{3^n}$ is a geometric series with $r = \frac{1}{3}$, hence it converges to its sum, $S_\infty = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} = 1.5$.

c $\lim_{n\to\infty} \frac{2n!}{3n!+3} = \lim_{n\to\infty} \frac{2}{3+\frac{3}{n!}} = \frac{2}{3} \neq 0$, hence the series diverges.

d $\sum_{n=0}^{\infty} 3\cdot\left(\frac{1}{7}\right)^{n-1} = 3\cdot 7 + \sum_{n=1}^{\infty} 3\cdot\left(\frac{1}{7}\right)^{n-1} \cdot \sum_{n=1}^{\infty} 3\cdot\left(\frac{1}{7}\right)^{n-1}$

is a geometric series with $r = \frac{1}{7}$, hence

converges to its sum, $S_\infty = 3\cdot\frac{1}{1-\frac{1}{7}} = \frac{21}{6} = 3\frac{1}{2}$.

Hence $S = 24.5$

e $\sum_{n=1}^{\infty} \frac{e^{n\pi}}{\pi^{ne}} = \sum_{n=1}^{\infty}\left(\frac{e^{\pi}}{\pi^{e}}\right)^n$ is a geometric series with

$r = \frac{e^{\pi}}{\pi^{e}} \approx 1.03$. Since $r > 1$ the series does not converge.

f $\frac{3}{2} + \frac{6}{3} + \frac{9}{4} + \frac{12}{5} \ldots = \sum_{n=1}^{\infty} \frac{3n}{n+1} . \lim_{n\to\infty} \frac{3n}{n+1}$

$= \lim_{n\to\infty} \frac{3}{1+\frac{1}{n}}$

$= 3 \neq 0$, hence series diverges.

g $\sum_{n=1}^{\infty} \frac{3}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{6}{2^n} = 6\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \sum \frac{1}{2^n}$ is an infinite geometric series, $r = \frac{1}{2}$, hence

$S_\infty = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1 \Rightarrow S_\infty = 6.$

h $\sum_{n=0}^{\infty}\left(\frac{1}{2^n} - \frac{1}{3^n}\right) = \sum_{n=0}^{\infty}\frac{1}{2^n} - \sum_{n=0}^{\infty}\frac{1}{3^n}$. Both series are geometric with $r = \frac{1}{2}$ and $r = \frac{1}{3}$ respectively.

Hence $S_\infty = \frac{1}{1-\frac{1}{2}} - \frac{1}{1-\frac{1}{3}} = 2 - \frac{3}{2} = 0.5.$

i $\lim_{n\to\infty} \frac{2^n}{2^{n+1}+1} = \lim_{n\to\infty} \frac{2^n}{2\cdot 2^n + 1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{2^n}} = \frac{1}{2} \neq 0,$

hence the series diverges by the nth term test.

2 $\frac{1}{2(n-1)} - \frac{1}{2(n+1)} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$

$= \frac{1}{2}\left(\frac{(n+1)-(n-1)}{n^2-1}\right) = \frac{1}{2}\left(\frac{2}{n^2-1}\right) = \frac{1}{n^2-1}.$

$\sum_{n=2}^{\infty} \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$

$= \frac{1}{2}\left(\left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right)\right.$

$\left. + \left(\frac{1}{4} - \frac{1}{6}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+2}\right) + \ldots\right)$

$= \frac{1}{2}\left(1 + \frac{1}{2} + \lim_{n\to\infty}\left(\frac{1}{n} + \frac{1}{n+2}\right)\right) = \frac{3}{4}$

3 $3 + \frac{9}{x} + \frac{27}{x^2} + \ldots = \sum_{n=1}^{\infty} \frac{3^n}{x^{n-1}}$. This is a geometric series with $r = \frac{3}{x}$. For the series to converge,

$\left|\frac{3}{x}\right| < 1 \Rightarrow x < -3,\ x > 3.\ S_\infty = \frac{3}{1-\frac{3}{x}} = \frac{3x}{x-3}.$

Exercise 4B

1. Corollary 1: $\int_x^a f(t)\,dt = -\int_a^x f(t)\,dt$

By the FTC, $g(x) = \int_x^a f(t)\,dt,\ b \le x \le a$ is an anti-derivative of f, i.e., $g'(x) = f(x),\ b < x < a$

$= -\int_x^a f(t)\,dt.$

Corollary 2: $\int_a^b f(x)\,dx = F(a) - F(b)$

Let $g(x) = \int_a^x f(t)\,dt,\ a \le x \le b$. If furthermore F is any anti-derivative of f, then by the mean value theorem, $F(x) = g(x) + c$, for some constant c. Evaluating therefore $F(b) - F(a)$,

$F(b) - F(a) = [g(b) + c] - [g(a) + c]$

$= g(b) - g(a)$

$= \int_a^b f(t)\,dt - \int_a^a f(t)\,dt$

$= \int_a^b f(t)\,dt - 0$

$= \int_a^b f(t)\,dt$

2 $\sum_{i=1}^{n} f(c_i \Delta x_i) = \sum_{i=1}^{n} \sin(c_i)\frac{\pi}{n} = \sum_{i=1}^{n} \sin\left(\frac{i-1}{n}\right)\frac{\pi}{n}$

$= \frac{\pi}{n}\left(\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{3\pi}{n}\right)\right.$

$\left. + \ldots + \sin\left(\frac{(n-1)\pi}{n}\right)\right)$

Hence,

$\pi \lim_{n\to\infty} \frac{\pi}{n}\left(\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right)\right.$

$\left. + \sin\left(\frac{3\pi}{n}\right) + \ldots + \sin\left(\frac{(n-1)\pi}{n}\right)\right)$

$= \int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi} = 2$

And therefore,

$\lim_{n\to\infty} \frac{\pi}{n}\left(\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right)\right.$

$\left. + \sin\left(\frac{3\pi}{n}\right) + \ldots + \sin\left(\frac{(n-1)\pi}{n}\right)\right)$

$= \frac{2}{\pi}$

3 **i** $\Delta x = \frac{5-0}{n} = \frac{5}{n};\ x_i = 0 + i\frac{5}{n} = \frac{5i}{n};$

$f(x_i) = 2 \cdot \frac{5i}{n} = \frac{10i}{n}$

$f(x_i)\Delta x = \sum_{i=1}^{n}\left(\frac{10i}{n}\right)\left(\frac{5}{n}\right) = \sum_{i=1}^{n}\frac{50i}{n^2} = \frac{50}{n^2}\sum_{i=1}^{n} i$

$= \frac{50}{n^2}\left(\frac{n(n+1)}{2}\right) = 25 \cdot \frac{n+1}{n}$

$\int_0^5 2x \, dx = \lim_{n\to\infty}\left(25 \cdot \frac{n+1}{n}\right) = 25\lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 25$

ii $\Delta x = \frac{0-(-2)}{n} = \frac{2}{n};$

$x_i = -2 + i\frac{2}{n};\ f(x_i) = 3\left(-2 + \frac{2i}{n}\right)^2 + 2\left(-2 + \frac{2i}{n}\right)$

$= \left(12 - \frac{24i}{n} + \frac{12i^2}{n^2}\right) - 4 + \frac{4i}{n} = 8 - \frac{20i}{n} + \frac{12i^2}{n^2}$

$f(x_i)\Delta x = \sum_{i=1}^{n}\left(\frac{16}{n} - \frac{40i}{n^2} + \frac{24i^2}{n^3}\right)$

$= \frac{16}{n} - \frac{40}{n^2}\sum_{i=1}^{n} i + \frac{24}{n^3}\sum_{i=1}^{n} i^2$

$= \frac{16}{n^3} - \frac{40}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{24}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$

$\int_{-2}^{0}(3x^2 + 2x)$

$= \lim_{n\to\infty}\left(\frac{16}{n} - \frac{40}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{24}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right) = 4$

4 **a** $f'(x) = \sqrt{3 + 4x}$

b $\int_x^4 2\tan t \, dt = -\int_4^x 2\tan t \, dt = -2\tan x$

c $\int_t^0 t\sec t \, dt = -\int_0^t t\sec t \, dt = -x\sec x$

5 **a** $F(4) = \int_4^4 \sqrt{t^2 + 9t} \, dt = 0$

b $F'(x) = \sqrt{x^2 + 9};\ F'(4) = \sqrt{25} = 5$

c $F''(x) = \frac{x}{\sqrt{x^2 + 9}};\ F''(4) = \frac{4}{5}$

6 $G'(x) = \sin x$

7 The curve will be concave up at $f''(x) > 0$.

$y' = \frac{1}{1 + x + x^2}$

$y''(x) = \frac{-(1 + 2x)}{(1 + x + x^2)^2}$

$y'' > 0 \Rightarrow -(1 + 2x) > 0,\ x < -\frac{1}{2}$

Exercise 4C

1 $\int_0^{\infty} e^{-x} dx = \lim_{b\to\infty}\int_0^b e^{-x} dx = \lim_{b\to\infty} -e^{-x}\Big|_0^b$

$= \lim_{b\to\infty}\left(-\frac{1}{e^b} + 1\right) = 1,$

hence the integral converges to 1.

2 $\int_0^{\infty} \sin x \, dx = \lim_{b\to\infty}\int_0^b \sin x \, dx = \lim_{b\to\infty}(-\cos x)\Big|_0^b$

$= \lim_{b\to\infty}(-\cos b + 1)$. Since $\cos b$ oscillates between -1 and 1, the integral diverges.

3 $\int_0^{\infty} \frac{e^x}{1+e^{2x}} dx = \lim_{b\to\infty}\int_0^b \frac{e^x}{1+e^{2x}} dx = \lim_{b\to\infty}\left[\arctan e^x\right]_0^b = \frac{\pi}{4},$

hence the integral converges to $\frac{\pi}{4}$.

4 $\int_1^{\infty} \frac{1}{\sqrt{x}} dx = \lim_{b\to\infty}\int_1^b \frac{1}{\sqrt{x}} dx = \lim_{b\to\infty}(2\sqrt{x})\Big|_1^b = \infty,$

hence the integral diverges.

5 $\int_0^{\infty} \cos \pi x \, dx = \lim_{b\to\infty}\int_0^b \cos \pi x \, dx$

$= \pi \lim_{b\to\infty} \sin x\Big|_0^b\ \pi\lim_{b\to\infty}(\sin b - 1)$. Since $\sin b$ oscillates between -1 and 1, the integral diverges.

6 $\int_0^{\infty} x^2 e^{-x} dx = \lim_{b\to\infty}\int_0^b x^2 e^{-x} dx$. Using integration by parts, $\lim_{b\to\infty}\int_0^b x^2 e^{-x} dx = \lim_{b\to\infty}\left[-e^{-x}\left(x^2 - 2x - 2\right)\right]_0^b = 2,$

hence the integral converges.

Exercise 4D

1 From the GDC, $f(x) = \dfrac{3}{x+1}$ is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.

$$\int_1^\infty \frac{3}{x+1}dx = \lim_{b\to\infty}\int_1^b \frac{3}{x+1}dx = 3\lim_{b\to\infty}\left[\ln(x+1)\right]_1^b$$

$$= 3\lim_{n\to\infty}(\ln(b+1) - \ln 2) = 3\lim_{n\to\infty}\ln\frac{b+1}{4} = \infty$$

hence the series diverges.

2 From the GDC, the function is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.

$$\int_1^\infty \frac{x}{x^2+1}dx = \lim_{b\to\infty}\int_1^b \frac{x}{x^2+1}dx = \lim_{b\to\infty} = \frac{\ln(x^2+1)}{2}\bigg|_1^b$$

$$= \frac{1}{2}\lim_{b\to\infty}\left[\ln(b^2+1) - \ln 2\right] = \infty,$$

hence the series diverges.

3 From the GDC, the function is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.

$$\int_1^\infty \frac{1}{x^2+1}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2+1}dx = \lim_{b\to\infty}\arctan x\Big|_1^b$$

$$= \lim_{b\to\infty}\left[\arctan b - \arctan 1\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

hence the series converges.

4 The integral test cannot be applied since the function is increasing for $x \geq 1$. However, since $\lim_{b\to\infty} x\sin\left(\dfrac{1}{x}\right) = 1 \neq 0$, the series diverges by the nth test for divergence.

5 From the GDC, the function is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.

$$\int_0^\infty \frac{e^x}{1+e^{2x}}dx = \lim_{b\to\infty}\int_0^b \frac{e^x}{1+e^{2x}}dx = \lim_{b\to\infty}\left[\arctan e^x\right]_0^b = \frac{\pi}{2}$$

Hence, the series converges.

6 The integral test cannot be applied since the function is not decreasing for all $x \geq 1$.

7 From the GDC, the function is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.

$$\int_1^\infty \frac{4x}{(2x^2+3)^2}dx = \lim_{b\to\infty}\int_1^b \frac{4x}{(2x^2+3)^2}dx = \lim_{b\to\infty}\left[-\frac{1}{2x^2+3}\right]_1^b$$

$$= \lim_{b\to\infty}\left(-\frac{1}{2b^2+3} + \frac{1}{5}\right) = \frac{1}{5}$$

Hence, the series converges.

8 The integral test cannot be applied since the function is not decreasing for all $x \geq 1$.

9 The integral test cannot be applied since the function is not decreasing for all $x \geq 1$.

10 From the GDC, the function is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.

$$\int_1^\infty \frac{1}{x\ln x}dx = \lim_{b\to\infty}\int_1^b \frac{1}{x\ln x}dx = \lim_{b\to\infty}\ln(\ln(x))\Big|_1^b$$

$$= \lim_{b\to\infty}\left(\text{In}(\text{In}(b) - \text{In}(\text{In}(1)\right) = \infty$$

Hence the series diverges.

11 From the GDC, the function is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.

$$\int_1^\infty \frac{\arctan x}{x^2+1} = \lim_{b\to\infty}\int_1^b \frac{\arctan x}{x^2+1} = \lim_{b\to\infty}\left[\frac{(\arctan x)^2}{2}\right]_1^b = \frac{3\pi^2}{32}$$

Hence the series converges.

12 From the GDC, the function is continuous, positive and decreasing for all $x \geq 1$, hence we can apply the integral test.

$$\int_1^\infty \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\left(-\frac{\ln x+1}{x}\right)\Bigg|_1^b$$

$$= \lim_{b\to\infty}\left(-\frac{\ln b+1}{b} + \frac{\ln 2}{2}\right) = \frac{\ln 2}{2}$$

Hence the series converges.

13 From the GDC, the function is continuous, positive and decreasing for all $x \geq 2$, hence we can apply the integral test.

$$\int_1^\infty \frac{x^2}{e^x}dx = \lim_{b\to\infty}\int_1^b \frac{x^2}{e^x}dx = \lim_{b\to\infty}(-e^{-x}(x^2+2x+2))\Big|_1^b$$

$$= \lim_{b\to\infty}\left((-e^{-b}(b^2+2b+2)) + \frac{5}{e}\right) = \frac{5}{e}.$$

Hence the series converges.

Exercise 4E

1 Using the p-series test, $p = \pi$, $\pi > 1$, hence the series converges.

2 Using the p-series test, $p = \dfrac{\pi}{4}$, $\dfrac{\pi}{4} < 1$, hence the series diverges.

3 $1 + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \dfrac{1}{4\sqrt{4}} + \ldots = \displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}$

Using the p-series test, $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$, $p = \dfrac{3}{2}$, $\dfrac{3}{2} > 1$,

hence the series converges.

4 $1 + \dfrac{1}{\sqrt[5]{4}} + \dfrac{1}{\sqrt[5]{9}} + \dfrac{1}{\sqrt[5]{16}} + \ldots = \displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt[5]{n^2}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{2}{5}}}.$

Using the p-series test, $p = \dfrac{2}{5}$, $\dfrac{2}{5} < 1$ hence the series diverges.

5 $p = \dfrac{1}{2}$, $\dfrac{1}{2} < 1$, hence the series diverges.

Exercise 4F

1 $\frac{1}{n^3+1} < \frac{1}{n^3}$, $n \ge 1$. Since $\frac{1}{n^3}$ converges by the p-series test, the original series also converges.

2 $\frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n}-1}$, $n \ge 2$. Since $\frac{1}{\sqrt{n}}$ diverges by the p-series test, the original series also diverges.

3 $\frac{1}{(n+1)!} < \frac{1}{n^2}$, $n \ge 1$. Since $\frac{1}{n^2}$ converges by the p-series test, the original series converges.

4 $\ln(n) \le n$ for $n \ge 2$, hence $\frac{1}{\ln n} \ge \frac{1}{n}$, $n \ge 2$. Since $\frac{1}{n}$ diverges by the p-series test, the original series diverges.

5 $\frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}}$; $n \ge 1$. Since $\frac{1}{\sqrt{n^3}}$ converges by the p-series test, the original series converges.

6 $\ln(n) > 1$ for $n \ge 3$, hence $\frac{\ln n}{n} \ge \frac{1}{n}$ for $n \ge 3$. Since $\frac{1}{n}$ diverges by the p-series test, the original series diverges.

7 $\frac{\cos^2 n}{n^2} \le \frac{1}{n^2}$ for $n \ge 1$. Since $\frac{1}{n^2}$ converges by the p-series test, the original series converges.

8 $\frac{n-1}{n^2\sqrt{n}} = \frac{n-1}{n^{\frac{5}{2}}}$; $\frac{n-1}{n^{\frac{5}{2}}} < \frac{1}{n^{\frac{5}{2}}}$ for $n \ge 1$. Since $\frac{1}{n^{\frac{5}{2}}}$ converges by the p-series test, the original series converges.

9 $\arctan x < \frac{\pi}{2}$; $\frac{1}{\sqrt{n^3+1}} < \frac{1}{\sqrt{n^3}}$ for $n \ge 1$. Hence

$\sum \frac{\arctan n}{\sqrt{n^3+1}} < \frac{\pi}{2}\sum \frac{1}{\sqrt{n^3}}$. Since $\frac{1}{\sqrt{n^3}}$ converges by the p-series test, the original series also converges.

10 $\ln(n) \ge 2$ for $n \ge e^2$ hence $\frac{1}{n^{\ln n}} \le \frac{1}{n^2}$. Since $\frac{1}{n^2}$ converges by the p-series test, the original series converges.

Exercise 4G

1 Compare with $\sum \frac{1}{\sqrt{n^2}}$. Hence,

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{\sqrt{n^2}}} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n^2}}} = 1,$$ and the series diverges since $\sum \frac{1}{\sqrt{n^2}} = \sum \frac{1}{n}$ diverges.

2 Compare with $\sum \frac{n^2}{n^5}$. Hence,

$$\lim_{n\to\infty} \frac{\frac{2n^2-1}{4n^5+n-1}}{\frac{n^2}{n^5}} = \lim_{n\to\infty} \frac{2n^7-2n^2}{4n^7+n^6-n^5} = \frac{1}{2}.$$ The series converges since by the p-series test, $\sum \frac{n^2}{n^5} = \sum \frac{1}{n^3}$ converges.

3 Compare with $\sum \frac{1}{n}$. Hence $\lim_{n\to\infty} \frac{\frac{1}{n+\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+\sqrt{n}} = 1$.

The series diverges, since $\sum \frac{1}{n}$ diverges.

4 Compare with $\sum \frac{1}{2^n}$, a convergent geometric series.

Since $\lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n-1} = 1$, the series converges.

5 Compare with $\sum \frac{n^2}{\sqrt{n^5}}$.

$$\lim_{n\to\infty} \frac{\frac{3n^2+2n}{\sqrt{4+n^5}}}{\frac{n^2}{\sqrt{n^5}}} = \lim_{n\to\infty}\left(\frac{3n^2+2n}{n^2} \times \sqrt{\frac{n^5}{4+n^5}}\right) = 3.$$

Since $\sum \frac{n^2}{\sqrt{n^5}} = \sum \frac{1}{\sqrt{n}}$, the series diverges.

6 $\sum\left(\frac{1}{2^n-1} - \frac{1}{2^n}\right) = \sum \frac{1}{4n^2-2n}$. Compare with $\sum \frac{1}{n^2}$.

$$\lim_{n\to\infty} \frac{\frac{1}{4n^2-2n}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{4n^2-2n} = \frac{1}{4}.$$ Hence, since $\sum \frac{1}{n^2}$ converges by the p-series test, the series converges.

7 Compare with $\sum \frac{1}{\sqrt{n}}$.

$$\lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}+\sqrt{n+2}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n}+\sqrt{n+2}} = \frac{1}{3}.$$ Hence, since $\sum \frac{1}{\sqrt{n}}$ diverges by the p-series test, the series diverges.

8 Compare with $\sum \frac{n2^n}{n^2} = \sum \frac{2^n}{n}$, which diverges by the nth term test for divergence, since $\lim_{n\to\infty} \frac{2^n}{n^2} \ne 0$.

$$\lim_{n\to\infty} \frac{\frac{n2^n+1}{3n^2+2n}}{\frac{2^n}{n}} = \lim_{n\to\infty} \frac{n^2 2^n + n}{3n^2 2^n + 2n2^n} = \lim_{n\to\infty} \frac{1+\frac{1}{n2^n}}{3+\frac{2}{n}} = \frac{1}{3}.$$

Hence the series diverges.

Exercise 4H

1 $$\lim_{n\to\infty} \frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\cdot\frac{2^{n+1}}{2^n}\right) = 2\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 2.$$

Since $L > 1$ the series diverges.

2 $$\lim_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Since $L < 1$ the series converges.

3 $$\lim_{n\to\infty} \frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \lim_{n\to\infty}\left(\frac{3^{n+1}}{3^n}\cdot\frac{n!}{(n+1)!}\right) = 3\lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Since $L < 1$ the series converges.

4 $\lim_{n\to\infty}\frac{\frac{3}{(n+1)2^{n+1}}}{\frac{3}{n2^n}} = \lim_{n\to\infty}\frac{n2^n}{(n+1)2^{n+1}} = \frac{1}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{1}{2}$.

Since $L < 1$ the series converges.

5 $\lim_{n\to\infty}\frac{\frac{\pi^{n+1}}{3(n+1)+2}}{\frac{\pi^n}{3n+2}} = \pi\lim_{n\to\infty}\frac{3n+2}{3n+5} = \pi$.

Since $L > 1$ the series diverges.

6 $\lim_{n\to\infty}\frac{\frac{5^{n+3}}{(n+2)!}}{\frac{5^{n+2}}{(n+1)!}} = 5\lim_{n\to\infty}\frac{(n+1)!}{(n+2)!} = 5\lim_{n\to\infty}\frac{1}{n+2} = 0$.

Since $L < 1$ the series converges.

Exercise 4I

1 Using the ratio test,

$$\lim_{n\to\infty}\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty}\left(\frac{2^n}{2^{n+1}}\cdot\frac{n+1}{n}\right) = \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{1}{2}.$$

Since $L < 1$ the series converges by the ratio test. Since the series converges absolutely, the series converges.

2 $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ has terms of different signs, since $-1 \le \sin n \le 1$. Since $\left|\frac{\sin n}{n^2}\right| \le \frac{1}{n^2}$ for all n, and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges by the p-series test, the series converges absolutely, therefore the series converges.

3 Using the ratio test,

$$\lim_{n\to\infty}\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = 2\lim_{n\to\infty}\frac{n!}{(n+1)!} = 2\lim_{n\to\infty}\frac{1}{n+1} = 0.$$

Since $L < 1$ the series converges by the ratio test. Since the series converges absolutely, the series converges.

4 Using the comparison test, $\lim_{n\to\infty}\frac{\arctan n}{n^3} \le \frac{\frac{\pi}{2}}{n^3}$ for all $n \ge 1$. Since $\sum\frac{1}{n^3}$ converges by the p-series test, so does $\frac{\pi}{2}\sum\frac{1}{n^3}$. Hence the series converges absolutely.

5 Using the ratio test,

$$\lim_{n\to\infty}\frac{\frac{1}{2^{n+1}(n+1)!}}{\frac{1}{2^n n!}} = \lim_{n\to\infty}\frac{2^n n!}{2^{n+1}(n+1)!} = \frac{1}{2}\lim_{n\to\infty}\frac{1}{n+1} = 0.$$

Since $L < 1$, the series converges. Hence the series converges absolutely.

6 Since $\lim_{n\to\infty}\frac{n!}{e^n} \ne 0$, the series does not converge absolutely.

Exercise 4J

1 a The absolute value of the terms is decreasing, and $\lim_{n\to\infty} u_n = 0$, hence the series converges. For the series of absolute values, $\ln(n) \le n$ for $n \ge 2$, hence $\frac{1}{\ln n} \ge \frac{1}{n}$, $n \ge 2$. Since $\frac{1}{n}$ diverges by the p-series test, the series of absolute values diverges by the comparison test. Hence the series converges conditionally.

b The absolute value of the terms is decreasing, and $\lim_{n\to\infty} u_n = 0$, hence the series converges. Using the ratio test for the series of absolute values,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty}\left[\frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n}\right]$$
$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\cdot\frac{1}{2}\right] = \frac{1}{2}.$$

Since $L < 1$, the series converges absolutely.

c The absolute value of the terms is decreasing, and $\lim_{n\to\infty} u_n = 0$, hence the series converges. The series of absolute values diverges by the p-series test, hence the series converges conditionally.

d The absolute value of the terms is decreasing, and $\lim_{n\to\infty} u_n = 0$, hence the series converges. For the series of absolute values, using the ratio test,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0.$$

Since $L < 1$, the series converges absolutely.

e The absolute value of the terms is decreasing, and $\lim_{n\to\infty} u_n = 0$, hence the series converges. The series of absolute values converges because it is a geometric series, with $|r| < 1$. Hence the series converges absolutely.

f $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \ldots + (-1)^n\frac{1}{n} + \ldots$

The absolute value of the terms is decreasing, and $\lim_{n\to\infty} u_n = 0$, hence the series converges. The series of absolute values is the harmonic series, which diverges. Hence the series converges conditionally.

g The absolute value of the terms is decreasing, and $\lim_{n\to\infty} u_n = 0$, hence the series converges. The integral test can be used on the series of absolute values, since $f(x) = \frac{1}{x\ln x}$ is continuous, positive and decreasing for all $x \ge 1$.

Hence, $$\int_1^{\infty}\frac{1}{x\ln x} = \lim_{b\to\infty}\int_1^b\frac{1}{x\ln x} = \lim_{n\to\infty}\left[\ln(\ln x)\right]_1^b$$
$$= \lim_{n\to\infty}\left[\ln(\ln b) - \ln(\ln 1)\right] = \infty.$$

Since the series of absolute values diverges, the series converges conditionally.

h The absolute value of the terms is decreasing, and $\lim_{n\to\infty} u_n = 0$, hence the series converges. The series of absolute values converges by comparison with $\sum\frac{\frac{\pi}{2}}{n^2}$, hence the series converges absolutely.

2 The conditions for Leibniz' theorem are met since the absolute value of the terms is decreasing, and $\lim_{n\to\infty} u_n = 0$. Hence the truncation error is $u_{11} = \frac{1}{11^3} = \frac{1}{1331}$. $S = S_{10} + \frac{1}{1331} \approx 0.902$.

3 The conditions for Leibniz' theorem are met since the absolute value of the terms is decreasing, and $\lim_{n\to\infty} u_n = 0$, hence the series converges.

$u_{n+1} \le 0.001 \Rightarrow \frac{1}{(n+1)^4} \le 0.001 \Rightarrow n = 5$. Hence, $S_5 \approx 0.948$.

4 The conditions for Leibniz' theorem are met since the absolute value of the terms is decreasing, and $\lim_{n\to\infty} u_n = 0$, hence the series converges. Since the series begins with $n = 2$, and we want the first three terms, i.e., $n = 2, 3, 4$, therefore $u_5 = \frac{1}{5^3 \times 3^5} = \frac{1}{30375}$. $S = S_3 + u_5 = -0.012644$.

5 The conditions for Leibniz' theorem are met since the absolute value of the terms is decreasing, and $\lim_{n\to\infty} u_n = 0$.

$$u_{n+1} \le 10^{-5} \Rightarrow \frac{1}{(n+1)^3 \ln(n+1)} < 10^{-5} \Rightarrow n = 30.$$

Hence, the truncated series has the required accuracy after the 29th term, since the first term corresponds to $n = 2$, and not $n = 1$.

Exercise 4K

1 Using the ratio test,

$$\lim_{\to\infty} \frac{\frac{6^{n+1}}{(n+1)!}}{\frac{6^n}{n!}} = \lim_{n\to\infty}\left(\frac{6^{n+1}}{6^n}\cdot\frac{n!}{(n+1)!}\right)$$

$$= 6\lim_{n\to\infty}\frac{1}{n+1} = 6\cdot 0 = 0.$$

Since $L < 1$, the series converges.

2 Using the ratio test,

$$\lim_{n\to\infty} \frac{\frac{(n+1)2^{n+1}}{3^{n+1}}}{\frac{n2^n}{3^n}} = \lim_{n\to\infty}\left(\frac{2\cdot 2^n(n+1)}{3\cdot 3^n}\cdot\frac{3^n}{n2^n}\right) = \frac{2}{3}\lim_{n\to\infty}\frac{n+1}{n}$$

$$= \frac{2}{3}\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{2}{3}.$$

Since $L < 1$, the series converges.

3 This is a geometric series whose common ratio is $\frac{1}{e}$. Since $|r| < 1$, the series converges to its sum.

4 Using the GDC, we see that the function that represents the series is continuous, positive and decreasing for all $n \ge 1$. Hence, we can use the integral test. Using the substitution $u = \frac{1}{x}$,

$$\int_1^\infty \frac{1}{x^2}\tan\left(\frac{1}{x}\right)dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\tan\left(\frac{1}{x}\right)dx$$

$$= \lim_{n\to\infty}\ln\left(\frac{1}{\cos x}\right)_b^\infty = -\ln(\cos(1)).$$

Hence the series converges.

5 Using the limit comparison test with $\sum\frac{n}{n^2}$, or $\sum\frac{1}{n}$,

$$\lim_{n\to\infty}\frac{\frac{2n+1}{2n^2+3n-1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{2n^2+n}{2n^2+3n-1} = 1.$$ Since L is a positive real number, the series diverges since the harmonic series diverges.

6 Since the absolute value of the terms of the series is decreasing, and $\lim_{n\to\infty} u_n = 0$, the series converges.

For all $n \ge 1$, $\sqrt{3n} > \sqrt{3n-1}$, hence $\frac{1}{\sqrt{3n}} < \frac{1}{\sqrt{3n-1}}$.

Since $\sum\frac{1}{\sqrt{n}}$ diverges by the p-series test, so does $\frac{1}{\sqrt{3}}\sum\frac{1}{\sqrt{n}}$. Hence the series does not converge absolutely, and converges conditionally.

7 $\sum\frac{2}{\ln n^2} = \sum\frac{2}{2\ln n} = \sum\frac{1}{\ln n}$. Since $\frac{1}{n} < \frac{1}{\ln n}$ for all $n \ge 2$, the series diverges by comparison with the harmonic series.

8 Since $\lim_{n\to\infty}\frac{n(n+1)}{\sqrt{n^3+5n^2}} \ne 0$, the series diverges by the Divergence Test.

9 Since the absolute value of the terms of the series is decreasing, and $\lim_{n\to\infty} u_n = 0$, the series converges. Using the integral test, since the function representing the series is positive, decreasing and continuous for all $x \ge 2$,

$$\int_2^\infty \frac{1}{x\sqrt{\ln x}}dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\sqrt{\ln x}}dx$$

$$= \lim_{b\to\infty}\left[2\sqrt{\ln x}\right]_2^b$$

$$= \lim_{b\to\infty}\left(2\sqrt{\ln b} - 2\sqrt{\ln 2}\right) = \infty.$$

Hence, the series diverges absolutely, and converges conditionally.

10 Using the ratio test,

$$\lim_{n\to\infty}\frac{\frac{4^{n+1}(n+1)^2}{(n+1)!}}{\frac{4^n n^2}{n!}} = \lim_{n\to\infty}\left(\frac{4^{n+1}(n+1)^2}{(n+1)!}\times\frac{n!}{4^n n^2}\right)$$

$$= \lim_{n\to\infty}\frac{4(n+1)}{n^2} = 0.$$

Since $L < 1$, the series converges.

11 Using the ratio test,

$$\lim_{n\to\infty}\frac{\dfrac{(n+1)!}{2\cdot5\cdot8\cdot\ldots\cdot(3(n+1)+2)}}{\dfrac{n!}{2\cdot5\cdot8\cdot\ldots\cdot(3n+2)}}$$

$$=\lim_{n\to\infty}\left(\frac{(n+1)!}{n!}\cdot\frac{2\cdot5\cdot8\cdot\ldots\cdot(3n+2)}{2\cdot5\cdot8\cdot\ldots\cdot(3n+2)\cdot(3(n+1)+2)}\right)$$

$$=\lim_{n\to\infty}\frac{(n+1)}{3n+5}=\frac{1}{3}$$

Since $L<1$ the series converges.

12 $\sum_{n=1}^{\infty}\frac{(-2)^{2n}}{n^n}=\sum_{n=1}^{\infty}\frac{4^n}{n^n}$. Using the ratio test,

$$\lim_{n\to\infty}\frac{\dfrac{4^{n+1}}{(n+1)^{n+1}}}{\dfrac{4^n}{n^n}}=\lim_{n\to\infty}\left(\frac{4\cdot4^n}{(n+1)(n+1)^n}\times\frac{n^n}{4^n}\right)$$

$$=\lim_{n\to\infty}\frac{4n^n}{(n+1)(n+1)^n}=0.$$

Hence, the series diverges. (This one is best done with the Root Test, which is not on the syllabus.)

13 Using the limit comparison test with $\sum\frac{1}{n\sqrt{n}}$,

$$\lim_{n\to\infty}\frac{\dfrac{\sin\left(\frac{1}{n}\right)}{\sqrt{n}}}{\dfrac{1}{n\sqrt{n}}}=\lim_{n\to\infty}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}=1,$$ and $1>0$, hence the

series converges since $\sum\frac{1}{n\sqrt{n}}$ converges by the p-series test.

14 $0<\frac{\arctan n}{n^{\frac{3}{2}}}<\frac{\frac{\pi}{2}}{n^{\frac{3}{2}}}$. $\sum\frac{\frac{\pi}{2}}{n^{\frac{3}{2}}}=\frac{\pi}{2}\sum\frac{1}{n^{\frac{3}{2}}}$. Since this series converges by the p-series test, the original series also converges.

15 a The series is continuous and decreasing, but not positive for all $n\ge1$.

b The series does not have all non-negative terms.

c The series does not have all non-negative terms.

d i the sequence of absolute values is not decreasing.

ii the limit of the nth term as n toes to infinity does not equal 0.

16 a $\sum_{n=1}^{\infty}\frac{n}{(n+1)!}$.

$S_1=\frac{1}{2!}$; $S_2=\frac{1}{2!}+\frac{2}{3!}=\frac{5}{6}$; $S_3=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}=\frac{23}{24}$;

$S_4=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}=\frac{119}{120}$

$S_5=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\frac{5}{6!}=\frac{719}{720}$;

$S_6=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\frac{5}{6!}+\frac{6}{7!}=\frac{5039}{5040}$.

Conjecture: $S_n=\frac{(n+1)!-1}{(n+1)!}$

b Proof by Mathematical Induction:

$P_n: S_n=\frac{(n+1)!-1}{(n+1)!},\ n\in\mathbb{Z}^+$.

For $n=1$, show that $\frac{(n+1)!-1}{(n+1)!}=\frac{n}{(n+1)!}$.

LHS: $\frac{(1+1)!-1}{(1+1)!}=\frac{2!-1}{2!}=\frac{1}{2!}=$ RHS

Assume P_k is true, i.e., $S_k=\frac{(k+1)!-1}{(k+1)!},\ k\in\mathbb{Z}^+$.

We want to show that P_{k+1} is true, i.e.,

$S_{k+1}=\frac{(k+2)!-1}{(k+2)!},\ k\in\mathbb{Z}^+$.

$$S_{k+1}=S_k+u_{k+1}$$

$$=\frac{(k+1)!-1}{(k+1)!}+\frac{k+1}{(k+2)!}\text{ by assumption}$$

$$=\frac{(k+2)[(k+1)!-1]+k+1}{(k+2)!}$$

$$=\frac{(k+2)!-(k+2)+k+1}{(k+2)!}$$

$$=\frac{(k+2)!-1}{(k+2)!}$$

$$=P_{k+1}$$

Since P_n is true for $n=1$, and proven true for $n=k+1$ when $n=k$, $k\in\mathbb{Z}^+$, then by mathematical induction P_{k+1} is true.

b. Using the ratio test, $\lim_{n\to\infty}\left[\frac{(n+1)}{(n+2)!}\times\frac{(n+1)!}{n}\right]=0$.

Since $L<1$, the series converges.

*Finding the sum is best done after Chapter 5 using $f(x)=e^x$ and Taylor series.

Let S denote $\sum_{n=1}^{\infty}\frac{n}{(n+1)!}$. Then

$$S+e-2=S+\sum_{n=2}^{\infty}\frac{1}{n!}=S+\sum_{n=1}^{\infty}\frac{1}{(n+1)!}=\sum_{n=1}^{\infty}\frac{1}{n!}=e-1.$$

Hence, $S=1$.

Review Exercise

1 a Using the nth term test for divergence,

$$\lim_{n\to\infty}\left(\frac{n}{n+4}\right)^n=\lim_{n\to\infty}\left(\frac{1}{1+\frac{4}{n}}\right)^n=\frac{1}{e^4}\neq0,$$ hence the

series diverges.

b Using the limit comparison test with $\sum\frac{1}{n}$,

$$\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}}=1,$$ hence by comparison with the harmonic series, the series diverges.

c For $n\ge2$, and since $2<e$, $\ln(n)<n-1$. Hence, $\frac{1}{\ln n}>\frac{1}{n-1}\Rightarrow\sum_{n=2}^{\infty}\frac{1}{\ln n}>\sum_{n=2}^{\infty}\frac{1}{n-1}=\sum_{n=1}^{\infty}\frac{1}{n}$. Since the harmonic series diverges, by the comparison test the given series will also diverge.

d By the ratio test,
$$\lim_{n\to\infty}\left(\frac{n+1}{e^{n+1}}\times\frac{e^n}{n}\right)=\lim_{n\to\infty}\left(\frac{n+1}{n}\times\frac{e^n}{e^{n+1}}\right)=\frac{1}{e}<1.$$ Hence the series converges.

e Since $\cos n\pi=(-1)^n$, $\frac{(n+10)\cos n\pi}{n^{1.4}}=(-1)^n\frac{(n+10)}{n^{1.4}}$.

Since $\lim_{n\to\infty}\frac{(n+10)}{n^{1.4}}=0$ and the sequence is decreasing, the given series is convergent by the alternating series test.

2 By the ratio test, $\lim_{n\to\infty}\left(\frac{k}{(k+1)!}\times\frac{k!}{(k-1)}\right)=\lim_{n\to\infty}\frac{k}{k^2-1}=0.$

Hence the series converges. $\sum_{k=1}^{\infty}\frac{(k-1)}{k!}=\sum_{k=1}^{\infty}\frac{1}{k}.$

Furthermore,
$$S=\frac{1}{1!}-\frac{1}{1!}+\frac{2}{2!}-\frac{1}{2!}+\frac{3}{3!}-\frac{1}{3!}+\ldots$$
$$=1-\left(\frac{1}{1!}-\frac{1}{1!}\right)-\left(\frac{1}{2!}-\frac{1}{2!}\right)-\left(\frac{1}{3!}-\frac{1}{3!}\right)-\ldots$$
$$=1.$$

3 $\lim_{n\to\infty}\frac{1}{(n+1)^7}=0$, and for all n, $a_n\ge a_{n+1}$, hence the series converges by the alternating series test 4. $8^{-7}=0.000000476837$. The sum of the series therefore, correct to 6d.p., is
$$1-\frac{1}{2^7}+\frac{1}{3^7}-\ldots+\frac{1}{7^7}=0.992594$$

4 By comparison with the harmonic series,
$$\lim_{n\to\infty}\left(\frac{1+n}{1+n^2}\times\frac{n}{1}\right)=\lim_{n\to\infty}\frac{n+n^2}{1+n^2}=1.$$ Hence the series diverges by the limit comparison test.

5 **a** Since $\sum_{n=1}^{\infty}a_n$ converges, then $\lim_{n\to\infty}a_n=0\Rightarrow a_n<1$ for large n. Since $b_n>0\Rightarrow a_nb_n<b_n$.

b By comparison, and since $\sum b_n$ converges, then $\sum a_nb_n$ converges. Similarly, $b_n<1\Rightarrow b_n^2<b_n$, and hence $\sum b_n^2$ converges because $\sum b_n$ converges.

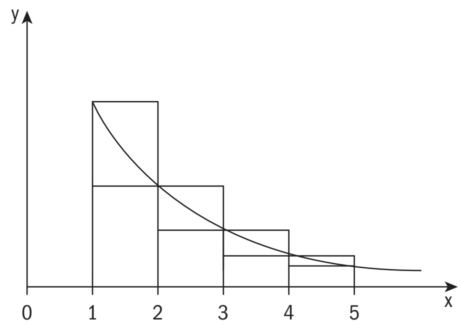
6 **a** $u_n=\frac{3n-2}{2^{\frac{n}{3}}}$. Using the ratio test,
$$\lim_{n\to\infty}\left|\frac{3n+1}{2^{\frac{n+1}{3}}}\times\frac{2^{\frac{n}{3}}}{3n-2}\right|=2^{\frac{1}{3}}\lim_{n\to\infty}\left(\frac{3n+1}{3n-2}\right)=2^{\frac{1}{3}}<1.$$
Hence the series converges.

b $u_n=\frac{2n-1}{\left(\sqrt{2}\right)^n}$. Then by the ratio test,
$$\lim_{n\to\infty}\left|\frac{2n+1}{\left(\sqrt{2}\right)^{n+1}}\times\frac{\left(\sqrt{2}\right)^n}{2n-1}\right|=\frac{1}{\sqrt{2}}\lim_{n\to\infty}\frac{2n+1}{2n-1}=\frac{1}{\sqrt{2}}<1.$$
Hence the series converges.

7 Using the limit comparison test with $b_n=\frac{1}{n^2}$,
$$\lim_{n\to\infty}\left|\frac{1-\cos\left(\frac{1}{n}\right)}{\frac{1}{n^2}}\right|=\frac{0}{0},$$ an indeterminate form which allows us to apply L'Hopital's rule. Taking the derivative of numerator and denominator, we obtain
$$\lim_{n\to\infty}\left|\frac{\sin\frac{1}{n}\times\frac{-1}{n^2}}{\frac{-2}{n^3}}\right|=\frac{1}{2}\lim_{n\to\infty}\left|\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right|=\frac{1}{2}\lim_{n\to 0}\frac{\sin n}{n}=\frac{1}{2}.$$

Since $\sum\frac{1}{n^2}$ converges by the p-series test, the given series also converges.

8 **a**



The sum of the areas of the smaller rectangles is $\sum_{n=2}^{\infty}\frac{1}{n^p}$.

The sum of the areas of the larger rectangles is $\sum_{n=1}^{\infty}\frac{1}{n^p}$.

Since the area under the curve is between smaller areas and the larger areas, $\sum_{n=2}^{\infty}\frac{1}{n^p}<\int_1^{\infty}\frac{1}{x^p}dx<\sum_{n=1}^{\infty}\frac{1}{n^p}$.

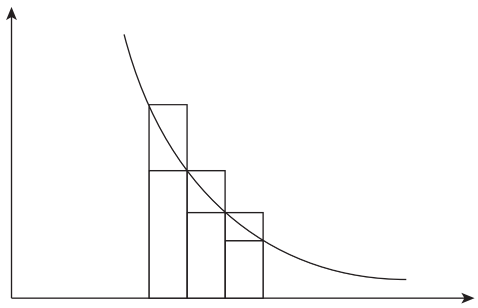
b $\int_1^{\infty}\frac{1}{x^p}dx=\left[\frac{x^{-p+1}}{-p+1}\right]_1^{\infty}=\frac{1}{p-1}.$

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=1+\sum_{n=2}^{\infty}\frac{1}{n^p}<1+\int_1^{\infty}\frac{1}{x^p}dx=1+\frac{1}{p-1}=\frac{p}{p-1}.$$

$\int_1^{\infty}\frac{1}{n^p}<\frac{p}{p-1}$ and $\int_1^{\infty}\frac{1}{x^p}dx<\sum_{n=1}^{\infty}\frac{1}{n^p}$.

Hence, $\frac{1}{p-1}<\sum_{n=1}^{\infty}\frac{1}{n^p}<\frac{p}{p-1}.$

9 a



The total area of the upper rectangles is

$$\frac{1}{n^4}\times 1+\frac{1}{(n+1)^4}\times 1+\frac{1}{(n+2)^4}\times 1+\ldots=\sum_{i=n}^{\infty}\frac{1}{i^4}.$$

The total area of the lower rectangles is

$$\frac{1}{(n+1)^4}\times 1+\frac{1}{(n+2)^4}\times 1+\frac{1}{(n+3)^4}\times 1+\ldots=\sum_{i=n+1}^{\infty}\frac{1}{i^4}.$$

The total area under the curve from $x = n$ to infinity lies between the above two sums, hence

$$\sum_{i=n+1}^{\infty}\frac{1}{i^4}<\int_n^{\infty}\frac{1}{x^4}dx<\sum_{i=n}^{\infty}\frac{1}{i^4}.$$

b $\int_n^{\infty}\frac{1}{x^4}dx=-\left[\frac{1}{3x^3}\right]_n^{\infty}=\frac{1}{3n^3}$. Hence, $\sum_{i=n+1}^{\infty}\frac{1}{i^4}<\frac{1}{3n^3}$.

Then, adding $\sum_{i=1}^{n}\frac{1}{i^4}$ to both sides of the inequality, $S<\sum_{i=1}^{n}\frac{1}{i^4}+\frac{1}{3n^3}$. In the same way $\sum_{i=n}^{\infty}\frac{1}{i^4}>\frac{1}{3n^3}$, and adding $\sum_{i=1}^{n-1}\frac{1}{i^4}$ to both sides, $S>\sum_{i=1}^{n-1}\frac{1}{i^4}+\frac{1}{3n^3}$. Hence, the value of S lies between $\sum_{i=1}^{n}\frac{1}{i^4}+\frac{1}{3n^3}$ and $\sum_{i=1}^{n-1}\frac{1}{i^4}+\frac{1}{3n^3}$.

c i When $n = 8$, $S < 1.08243\ldots$ and $S > 1.08219..$, hence $S = 1.082$ to 3 d.p.

ii Substituting this value for S,

$$N\approx\frac{\pi^4}{1.082}\approx 90.0268\ldots\Rightarrow N=90.$$

10 a $S_{2n}=S_n+\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}>S_n+\frac{1}{2n}+\frac{1}{2n}+\ldots+\frac{1}{2n}=S_n+\frac{1}{2}.$

b Replacing n by $2n$, $S_{4n}>S_{2n}+\frac{1}{2}>S_n+1$.

Continuing this way, $S_{8n}>S_n+\frac{3}{2}$. And in general, $S_{2^m n}>S_n+\frac{m}{2}$. Then, letting $n = 2$, $S_{2^{m+1}}>S_2+\frac{m}{2}$.

c $S_{2^{m+1}}>N$ if $S_2+\frac{m}{2}>N$, or if $m>2(N-S_2)$.
Hence the sequence is divergent.

11 i The sequence of absolute values is decreasing and the nth term tends to 0 as n tends to infinity, hence by the alternating series test, the series converges.

ii The partial sums are 0.333, 0.111, 0.269, 0.148, 0.246. S_∞ lies between any pair of successive partial sums, hence it lies between 0.148 and 0.246, therefore it is less than 0.25.

12 a The sequence of absolute values is decreasing and the nth term tends to 0 as n tends to infinity, hence by the alternating series test, the series converges.

b $S_4=1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}=0.841468(6\text{ d.p.}).$

S_4 $error<a_5$, hence

$error<\frac{1}{9!}$, or $error<0.00000276$.

5 Everything polynomic

Exercise 5A

a For $f(x) = \dfrac{1}{1-4x}$ $r = 4x$,

$|4x| < 1 \Rightarrow |x| < \dfrac{1}{4}$, $-\dfrac{1}{4} < x < \dfrac{1}{4}$, $x \in \left]-\dfrac{1}{4}, \dfrac{1}{4}\right[$.

In this interval,

$\dfrac{1}{1-4x} = 1 + 4x + (4x)^2 + \ldots + (4x)^n + \ldots = \displaystyle\sum_{n=0}^{\infty}(4x)^n$,

and $R = \dfrac{1}{4}$.

b For $f(x) = \dfrac{1}{1+5x}$, $r = -5x$; $|r| < 1 \Rightarrow |x| < \dfrac{1}{5}$,

$-\dfrac{1}{5} < x < \dfrac{1}{5}$, $x \in \left]-\dfrac{1}{5}, \dfrac{1}{5}\right[$. In this interval,

$$\frac{1}{1-(-5x)} = 1 + (-5x) + (-5x)^2 + (-5x)^3 + \ldots + (-5x)^n + \ldots$$

$$= 1 - 5x + 5^2x^2 + \ldots + (-1)^n(5x)^n + \ldots$$

Hence, $\dfrac{1}{1+5x} = \displaystyle\sum_{n=0}^{\infty}(-1)^n(5x)^n$; $R = \dfrac{1}{5}$.

c $f(x) = \dfrac{1}{x-4} = \dfrac{1}{4\left(\frac{1}{4}x - 1\right)} = \dfrac{1}{-4\left(1 - \frac{1}{4}x\right)}$.

$\left|\dfrac{1}{4}x\right| < 1 \Rightarrow |x| < 4$, $-4 < x < 4$, $x \in]-4, 4[$.

In this interval, $\dfrac{1}{x-4} = -\dfrac{1}{4}\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{4}x\right)^n$; $R = 4$

d $f(x) = \dfrac{1}{3x} = \dfrac{1}{3}\left(\dfrac{1}{x}\right) = \dfrac{1}{3}\left(\dfrac{1}{1+(x-1)}\right)$

$= \dfrac{1}{3}\left(\dfrac{1}{1-[-(x-1)]}\right)$; $|x-1| < 1 \Rightarrow 0 < x < 2$.

Hence, the interval is $\left]0, \dfrac{2}{3}\right[$

In this interval, $f(x) = \displaystyle\sum_{n=0}^{\infty}(-1)^n(x-1)^n$; $R = \dfrac{1}{3}$.

e $f(x) = \dfrac{x}{1+x}$; $|x < 1 \Rightarrow -1 < x < 1|$, $x \in]-1, 1[$. In this interval, $\dfrac{x}{1+x} = x(1 + (-x) + (-x)^2 + \ldots + (-x)^n + \ldots)$

$= x\displaystyle\sum_{n=0}^{\infty}(-1)^n x^n = \sum_{n=0}^{\infty}(-1)^n x^{n+1}$; $R = 1$

f $f(x) = \dfrac{2}{1+4x}$; $|-4x| < 1 \Rightarrow -\dfrac{1}{4} < x < \dfrac{1}{4}$, $x \in \left]-\dfrac{1}{4}, \dfrac{1}{4}\right[$.

In this interval,

$\dfrac{2}{1+4x} = 2(1 + (-4x) + (-4x)^2 + \ldots + (-4x)^n + \ldots)$

$= 2\displaystyle\sum_{n=0}^{\infty}(-1)^n(4x)^n = \sum_{n=0}^{\infty}2\cdot(-1)^n(4x)^n$; $R = \dfrac{1}{4}$.

Exercise 5B

a Using the ratio test, $\displaystyle\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{x^{n+1}}{n+1}}{(-1)^n\frac{x^n}{n}}\right| = |x|\lim_{n\to\infty}\frac{n}{n+1} = |x|$.

The series converges in the interval $|x| < 1$, hence $R = 1$. At $x = 1$, $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{n}$ is the alternating harmonic series, which converges conditionally. At $x = -1$, $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{(-1)^n}{n} = \sum_{n=0}^{\infty}\frac{(-1)^{2n}}{n} = \sum_{n=0}^{\infty}\frac{1}{n}$ is the harmonic series, which diverges. Hence, the interval of convergence is $]-1, 1[$, and $R = 1$.

b Using the ratio test,

$\displaystyle\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = |x|\lim_{n\to\infty}\frac{n!}{(n+1)!} = |x|\lim_{n\to\infty}\frac{1}{n+1} = 0.$

Hence, the interval of convergence is the set of Reals, and R = infinity.

c Using the ratio test,

$\displaystyle\lim_{n\to\infty}\left|\frac{\frac{2^{n+1}x^{n+1}}{(n+2)^2}}{\frac{2^n x^n}{(n+1)^2}}\right| = 2|x|\lim_{n\to\infty}\frac{(n+1)^2}{(n+2)^2} = 2|x|$. Hence, the series converges in the interval $2|x| < 1$, or $-\dfrac{1}{2} < x < \dfrac{1}{2}$. At $x = -\dfrac{1}{2}$, $\displaystyle\sum_{n=0}^{\infty}\frac{2^n\left(-\frac{1}{2}\right)^n}{(n+1)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}$. Since the absolute value of the series is decreasing, and $\displaystyle\lim_{n\to\infty}u_n = 0$, the series converges conditionally. At $x = \dfrac{1}{2}$, $\displaystyle\sum_{n=0}^{\infty}\frac{2^n\left(\frac{1}{2}\right)^n}{(n+1)^2} = \sum_{n=0}^{n}\frac{1}{(n+1)^2}$. This series converges by comparison with $\displaystyle\sum\frac{1}{n^2}$, which converges by the p-series test. Hence, the interval of convergence is $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$, and $R = \dfrac{1}{2}$.

d Using the ratio test, $\displaystyle\lim_{n\to\infty}\left|\frac{(-1)\frac{(2x+3)^{n+1}}{(n+1)\ln(n+1)}}{(-1)^n\frac{(2x+3)^n}{n\ln n}}\right|$

$= |2x+3|\displaystyle\lim_{n\to\infty}\left|\frac{n\ln n}{(n+1)\ln(n+1)}\right| = |2x+3|$. The interval of convergence is $|2x+3| < 1$, or $-2 < x < -1$. At $x = -2$, $\displaystyle\sum\frac{1}{n\ln n}$ is divergence by comparison with the harmonic series, and when $x = -1$, the series converges conditionally. Hence, the interval of convergence is $]-2, -1]$, and $R = \dfrac{1}{2}$.

e Using the ratio test, $\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+2}(x-1)^{n+2}}{n+2}}{\frac{(-1)^{n+1}(x-1)^{n+1}}{n+1}}\right|$

$= |-1(x-1)|\lim_{n\to\infty}\left|\frac{n+1}{n+2}\right| = |x-1|$. The interval of convergence is $|x-1|<1$, or $0<x<2$. At $x=0$, $\sum(-1)^{n+1}\left|\frac{(-1)^{n+1}}{n+1}\right| = \sum\frac{1}{n+1}$ diverges by comparison with $\frac{1}{\sqrt{n}}$, which diverges by the p-series test. At $x=2$, $\sum(-1)^{n+1}\frac{1}{n+1}$ converges conditionally by the alternating series test. Hence, the interval of convergence is $]0, 2]$, and $R = 1$.

f Using the ratio test, $\lim_{n\to\infty}\left|\frac{\frac{x^{2n+3}}{(2n+3)!}}{\frac{x^{2n+1}}{(2n+1)!}}\right| = x^2\lim_{n\to\infty}\frac{(2n+3)!}{(2n+1)!}$

$= x^2\lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)} = x^2\cdot 0 = 0$. Hence, the interval of convergence is the set of Reals, and R = infinity.

g Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)\sqrt{n+1}\cdot 3^n}}{\frac{x^n}{n\sqrt{n}\cdot 3^n}}\right| = |x|\lim_{n\to\infty}\frac{n\sqrt{n}}{(n+1)\sqrt{n+1}\cdot 3}$$

$$= \frac{|x|}{3}\lim_{n\to\infty}\frac{\sqrt{n^3}}{\sqrt{(n+1)^3}} = \frac{|x|}{3}.$$

The interval of convergence is $\frac{|x|}{3}<1 \Rightarrow |x|<3$.

At $x=3$, $\sum_{n=1}^{\infty}\frac{3n}{n\sqrt{n}\cdot 3^n} = \sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ converges by p-series test. At $x=-3$, $\sum_{n=1}^{\infty}\frac{(-3)^n}{n\sqrt{n}\cdot 3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^{\frac{3}{2}}}$, the series converges conditionally by the alternating series test. Hence the interval of convergence is $[-3,3]$, and $R = 3$.

h Using the ratio test, $\lim_{n\to\infty}\left|\frac{\frac{\sqrt{n+1}\cdot x^{n+1}}{3^{n+1}}}{\frac{\sqrt{n}\cdot x^n}{3^n}}\right| = \frac{|x|}{3}\lim_{n\to\infty}\frac{\sqrt{n+1}}{\sqrt{n}}$

$= \frac{|x|}{3}\lim_{n\to\infty}\sqrt{1+\frac{1}{\sqrt{n}}} = \frac{|x|}{3}$. The interval of convergence is $\frac{|x|}{3}<1 \Rightarrow |x|<3$. At $x=3$, $\sum_{n=1}^{\infty}\sqrt{n}$ diverges, and at $x=-3$ $\sum_{n=1}^{\infty}(-1)^n\sqrt{n}$ also diverges. Hence, the interval of convergence is $]-3, 3[$, and $R = 3$.

i Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{\frac{\sin^{n+1}x}{(n+1)^2}}{\frac{\sin x}{n^2}}\right| = |\sin x|\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = |\sin x|$$

The interval of convergence is $|\sin(x)|<1$, and this is true for all x. Hence, the interval of convergence is the set of Reals, and R = infinity.

j Using the ratio test, $\lim_{n\to\infty}\left|\frac{\frac{e^{(n+1)x}}{n+1}}{\frac{e^{nx}}{n}}\right| = e^x\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = e^x$

Hence, the interval of convergence is $e^x<1$, or $x<0$, and R = infinity.

k Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{\frac{e^{(n+1)x}}{n+1}}{\frac{e^{nx}}{n}}\right| = \lim_{n\to\infty}\left|\frac{x+1}{3}\right| = \frac{x+1}{3}.$$

This converges for $\left|\frac{x+1}{3}\right|<1 \Rightarrow -4<x<2$.

At $x=-4$, $\sum_{n=1}^{\infty}\frac{(-3)^n}{3} = \sum_{n=1}^{\infty}(-1)^n$, which diverges. At $x=2$, $\sum_{n=1}^{\infty}\frac{3^n}{3^n} = \sum_{n=1}^{\infty}1$ which also diverges. Hence, the interval of convergence is $]-4, 2[$, and $R = 3$.

2 **a** Applying the ratio test,

$$\lim_{n\to\infty}\left|\frac{\frac{(kx)^{n+1}}{(n+1)^2}}{\frac{(kx)^n}{n^2}}\right| = |x|\lim_{n\to\infty}\left|k\cdot\frac{n^2}{(n+1)^2}\right| = k|x|.$$

$k|x|<1 \Rightarrow |x|<\frac{1}{k}$. Since $x\in\left[-\frac{1}{2},\frac{1}{2}\right] \Rightarrow k=2$.

b Applying the ratio test, $\lim_{n\to\infty}\left|\frac{\frac{(x-k)^{2(n+1)}}{(n+1)^4}}{\frac{(x-k)^{2n}}{n^4}}\right|$

$= (x-k)^2\lim_{n\to\infty}\frac{n^4}{(n+1)^4} = (x-k)^2$. The interval of convergence is $(x-k)^2<1 \Rightarrow -1<x-k<1$ $\Rightarrow -1+k<x<1+k$. Since the interval of convergence is given as $[2, 4] \Rightarrow -1+k=2$ *and* $k+1=4$. Hence, $k=3$.

3 Applying the ratio test, $\lim_{n\to\infty}\left|\frac{\frac{(x-h)^{n+1}}{(n+1)k^{n+1}}}{\frac{(x-h)^n}{nk^n}}\right|$

$= \frac{|x-h|}{k}\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = \frac{|x-h|}{k}$. Since $\frac{|x-h|}{k}<1$

$\Rightarrow |x-h|<k \Rightarrow -k<x-h<k \Rightarrow -k+h<x<k+h$

Since the interval of convergence is $]-1, 7[\Rightarrow -k+h=1$ *and* $k+h=7$. Solving simultaneously, $h=4$ and $k=3$.

Exercise 5C

1 $\frac{2}{1-x^2} = 2(1+x^2+x^4+\ldots+x^{2n}+\ldots) = 2\sum_{n=0}^{\infty}x^{2n}$.

$I =]-1, 1[$, $R = 1$.

2 $\frac{1}{(x+1)^2} = \frac{1}{1+x}\cdot\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n\cdot\sum_{n=0}^{\infty}(-1)^n x^n$

$= \sum_{n=0}^{\infty}(-1)^n(n+1)x^n$. $I =]-1, 1[$; $R = 1$.

3 $\frac{1}{4x^2+1} = 1-4x^2+(4x^2)^2-(4x^2)^3+\ldots$

$= \sum_{n=0}^{\infty}(-1)^n(4x^2)^n = \sum_{n=0}^{\infty}(-1)^n(2x)^{2n}$.

$I = \left]-\frac{1}{2}, \frac{1}{2}\right[$; $R = \frac{1}{2}$.

4 This example should be done in 5F using the Binomial Series. The solution is shown here.

$$\sqrt{1+x} = (1+x)^{\frac{1}{2}}$$

$$= 1 + \frac{1}{2}x + \frac{1}{2}\times\left(-\frac{1}{2}\right)\frac{x^2}{2!} + \frac{1}{2}\times\left(-\frac{1}{2}\right)\times\left(-\frac{3}{2}\right)\frac{x^3}{3!} + \ldots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots; -1 < x < 1$$

$$\frac{1}{\sqrt{1-x}} = (1-x)^{-\frac{1}{2}}$$

$$= 1 - \left(-\frac{1}{2}\right)x + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^2}{2!} - \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{x^3}{3!} + \ldots$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \ldots, -1 < x < 1.$$

$$(1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

$$= \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{12}x^3 + \ldots\right)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{3x^4}{8}\ldots; \left]-1, 1\right[; R = 1$$

5 $\ln x = \int \frac{1}{x}; \frac{1}{x} = \frac{1}{1+(x-1)}$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \ldots$$

$$\int_1^x \frac{1}{1+(t-1)}dx = \int_1^x (1 - (t-1) + (t-1)^2 - (t-1)^3 + \ldots)dt$$

$$= \left[t - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} + \ldots\right]_1^x$$

$$= \left(x - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} + \ldots\right) - 1$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{(x-1)^{n+1}}{n+1}$$

$I = |x-1| < 1 \Rightarrow I = \left]0, 2\right[; R = 1$

6 $\sum_{n=0}^{\infty}(a_n x^n + b_n x^n)$ is convergent when $|x| > 1$, hence $R = 1$.

7 For $\sum_{n=1}^{\infty}\frac{x^n}{n^2 2^n}$, $\lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\times\frac{1}{2}x\right| = \left|\frac{x}{2}\right|$ which is convergent for $|x| < 2$, so $R = 2$. Testing the end-points, for $x = 2$, $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent by the p-series test. For $x = -2$, $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$ converges absolutely.

Hence, the interval of convergence is $[-2, 2]$

For $\sum_{n=1}^{\infty}\frac{nx^{n-1}}{n^2 2^n}$, $\lim_{n\to\infty}\left|\frac{n}{n+1}\times\frac{1}{2}x\right| = \left|\frac{x}{2}\right|$, which is also convergent for $|x| < 2$. Testing the end points, at $x = 2$, $\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, and at $x = -2$, $\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges conditionally.

Since $\frac{nx^{n-1}}{n^2 2^n}$ is the derivative of $\frac{x^n}{n^2 2^n}$, the interval of convergence would be the same, except for possibly the end points. Hence, the interval of convergence is $[-2, 2[$.

8 $\frac{2}{x+2} + \frac{1}{x-1} = \frac{2(x-1)+(x+2)}{(x+2)(x-1)} = \frac{3x}{x^2+x-2}$.

$$\frac{2}{x+2} = \sum_{n=0}^{\infty}\frac{1}{1-\left(-\frac{1}{2}x\right)} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}x\right)^n. I = \left]-2, 2\right[;$$

$R = 2$.

$$\frac{1}{x-1} = \sum_{n=0}^{\infty}\frac{-1}{1-x} = -1\sum_{n=0}^{\infty}\frac{1}{1-x} = \sum_{n=0}^{\infty}-x^n;$$

$I = \left]-1, 1\right[; R = 1$.

$$\frac{3x}{x^2+x-2} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}x\right)^n + \sum_{n=0}^{\infty}-x^n = \sum_{n=0}^{\infty}\left[\frac{1}{(-2)^n} - 1\right]x^n.$$

Hence, $I = \left]-1, 1\right[; R = 1$.

Exercise 5D

1 a $f(x) = \frac{1}{1-x} \Rightarrow f(0) = 1$,

$f'(x) = \frac{1}{(1-x)^2} \Rightarrow f'(0) = 1$,

$f''(x) = \frac{2}{(1-x)^3} \Rightarrow f''(0) = 2$, $f^{(3)}(x) = \frac{6}{(1-x)^4}$

$\Rightarrow f^{(3)}(0) = 6$,

$f^{(4)}(x) = \frac{24}{(1-x)^5} \Rightarrow f^{(4)}(0) = 24$

$$T_4(x) = \frac{1}{0!} + \frac{x}{1!} + \frac{2x^2}{2!} + \frac{6x^3}{3!} + \frac{24x^4}{4!}$$

$$= 1 + x + x^2 + x^3 + x^4$$

b $f(x) = x\sin x \Rightarrow f(0) = 0$,

$f'(x) = \sin x + x\cos x \Rightarrow f'(0) = 0$,

$f''(x) = \cos x + \cos x - x\sin x$

$= 2\cos x - x\sin x \Rightarrow f''(0) = 2$,

$f^{(3)}(x) = -2\sin x - \sin x - x\cos x$

$= -3\sin x - x\cos x \Rightarrow f^{(3)}(0) = 0$,

$f^{(4)}(x) = -3\cos x - \cos x + x\sin x$

$= -4\cos x + x\sin x \Rightarrow f^{(4)}(0) = -4$

$$T_4(x) = \frac{2x^2}{2!} + \frac{-4x^4}{4!} = x^2 - \frac{1}{6}x^4$$

c $f(x) = (1+x)^{\frac{1}{3}} \Rightarrow f(0) = 1$,

$f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}} \Rightarrow f'(0) = \frac{1}{3}$,

$f''(x) = -\frac{2}{9}(1+x)^{-\frac{5}{3}} \Rightarrow f''(0) = -\frac{2}{9}$,

$$f^{(3)}(x) = \frac{10}{27}(1+x)^{-\frac{8}{3}} \Rightarrow f^{(3)}(0) = \frac{10}{27},$$

$$f^{(4)}(x) = -\frac{80}{81}(1+x)^{-\frac{8}{3}} \Rightarrow f^{(4)}(0) = -\frac{80}{81}$$

$$T_4(x) = \frac{1}{0!} + \frac{\frac{1}{3}x}{1!} + \frac{-\frac{2}{9}x^2}{2!} + \frac{\frac{10}{27}x^3}{3!} + \frac{-\frac{80}{81}x^4}{4!}$$

$$= 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4$$

d $f(x) = xe^x \Rightarrow f(0) = 0,\ f'(x) = e^x + xe^x$

$\Rightarrow f'(0) = 1,$

$f''(x) = 2e^x + xe^x \Rightarrow f''(0) = 2,$

$f^{(3)}(x) = 3e^x + xe^x \Rightarrow f^{(3)}(0) = 3,$

$f^{(4)}(x) = 4e^x + xe^x \Rightarrow f^{(4)}(0) = 4$

$$T_4(x) = \frac{x}{1!} + \frac{2x^2}{2!} + \frac{3x^3}{3!} + \frac{4x^4}{4!} = x + x^2 + \frac{1}{2}x^3 + \frac{1}{6}x^4$$

2 a $f(x) = \frac{1}{x} \Rightarrow f(\pi) = \frac{1}{\pi},\ f'(x) = -\frac{1}{x^2}$

$\Rightarrow f'(\pi) = -\frac{1}{\pi^2},\ f''(x) = \frac{2}{x^3}$

$\Rightarrow f''(\pi) = \frac{2}{\pi^3},\ f^{(3)}(x) = -\frac{6}{x^4}$

$\Rightarrow f^{(3)}(\pi) = -\frac{6}{\pi^3}$

$$T_3(x) = \frac{\frac{1}{\pi}}{0!} + \frac{-\frac{1}{\pi^2}(x-\pi)}{1!} + \frac{\frac{2}{\pi^3}(x-\pi)^2}{2!} + \frac{-\frac{1}{\pi^2}(x-\pi)^3}{3!}$$

$$= \frac{1}{\pi} + -\frac{1}{\pi^2}(x-\pi) + \frac{2}{\pi^3}(x-\pi)^2 - \frac{1}{\pi^4}(x-\pi)^3$$

b $f(x) = x^{-1}\sin x \Rightarrow f(\pi) = 0,$

$f'(x) = -x^{-2}\sin x + x^{-1}\cos x \Rightarrow f'(\pi) = \frac{1}{\pi},$

$f''(x) = 2x^{-3}\sin x - x^{-2}\cos x - x^{-2}\cos x - x^{-1}\sin x$

$= 2x^{-3}\sin x - 2x^{-2}\cos x - x^{-1}\sin x$

$\Rightarrow f''(\pi) = -\frac{2}{\pi^2}$

$f^{(3)}(x) = -6x^{-4}\sin x + 2x^{-3}\cos x + 4x^{-3}\cos x$

$+ 2x^{-2}\sin x + x^{-2}\sin x - x^{-1}\cos x$

$= -6x^{-4}\sin x + 6x^{-3}\cos x + 3x^{-2}\sin x - x^{-1}\cos x$

$\Rightarrow f^{(3)}(\pi) = -\frac{6}{\pi^3} + \frac{1}{\pi}$

$$T_3(x) = \frac{-\frac{1}{\pi}(x-\pi)}{1!} + \frac{\frac{2}{\pi^2}(x-\pi)^2}{2!} + \frac{\left(-\frac{6}{\pi^3} + \frac{1}{\pi}\right)(x-\pi)^3}{3!}$$

$$= -\frac{1}{\pi}(x-\pi) + \frac{1}{\pi^2}(x-\pi)^2 + \left(\frac{1}{6\pi} - \frac{1}{\pi^3}\right)(x-\pi)^3$$

c $f(x) = x^2\cos x \Rightarrow f(\pi) = -\pi^2,$

$f'(x) = 2x\cos x - x^2\sin x \Rightarrow f'(\pi) = -2\pi,$

$f''(x) = 2\cos x - 2x\sin x - 2x\sin x - x^2\cos x$

$= (2 - x^2)\cos x - 4x\sin x$

$\Rightarrow f''(\pi) = \pi^2 - 2$

$f^{(3)}(x) = -2x\cos x - (2 - x^2)\sin x - 4\sin x - 4x\cos x$

$= -6x\cos x - (x^2 + 6)\sin x$

$\Rightarrow f^{(3)}(\pi) = -6\pi$

$$T_3(x) = \frac{-\pi^2}{0!} - \frac{2\pi(x-\pi)}{1!} + \frac{(\pi^2-2)(x-\pi)^2}{2!} + \frac{-6\pi(x-\pi)^3}{3!}$$

$$= -\pi^2 - 2\pi(x-\pi) + \left(\frac{\pi^2}{2} - 1\right)(x-\pi)^2 + \pi(x-\pi)^3$$

3 $f(x) = 2x^3 - 3x^2 + 4x - 5 \Rightarrow f'(x) = 6x^2 - 6x + 4$

$\Rightarrow f''(x) = 12x - 6 \Rightarrow f^{(3)}(x) = 12$

a $f(0) = -5,\ f'(0) = 4,\ f''(0) = -6,\ f^{(3)}(0) = 12$

$$\Rightarrow T_3(x) = \frac{-5}{0!} + \frac{4x}{1!} + \frac{-6x^2}{2!} + \frac{12x^3}{3!}$$

$$= -5 + 4x - 3x^2 + 2x^3$$

b $f(1) = 2 - 3 + 4 - 5 = -2,\ f'(1) = 6 - 6 + 4 = 4,$

$f''(1) = 12 - 6 = 6,\ f^{(3)}(1) = 12 \Rightarrow$

$$T_3(x) = \frac{-2}{0!} + \frac{4(x-1)}{1!} + \frac{6(x-1)^2}{2!} + \frac{12(x-1)^3}{3!}$$

$$= -2 + 4(x-1) + 3(x-1)^2 + 2(x-1)^3$$

4 $T_1(x) = \frac{f(a)}{0!} + \frac{f'(a)(x-a)}{1!} = f(a) + f'(a)(x-a)$

Equation of a tangent to the curve $y = f(x)$ at the point where $x = a$ is obtained in the following way: Point $(a, f(a))$, slope $m = f'(a)$

Equation: $y - y_1 = m(x - x_1) \Rightarrow y - f(a)$

$= f'(a)(x-a) \Rightarrow y = f(a) + f'(a)(x-a)$

5 $f(1) = 4,\ f'(1) = -1,\ f''(1) = 3,\ f^{(3)}(1) = 2 \Rightarrow$

$$T_3(x) = \frac{4}{0!} + \frac{-1(x-1)}{1!} + \frac{3(x-1)^2}{2!} + \frac{2(x-1)^3}{3!}$$

$$= 4 - (x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{4}(x-1)^3$$

$$f(1.2) \approx T_3(1.2) = 4 - 0.2 + \frac{3}{2} \times 0.2^2 + \frac{1}{4} \times 0.2^3$$

$$= 3.86267$$

Exercise 5E

1 $f(x) = \cos x \Rightarrow f'(x) = -\sin x$

$\Rightarrow f''(x) = -\cos x \Rightarrow f^{(3)}(x) = \sin x \Rightarrow f^{(4)}(x) = \cos x$

We can summarize this into the following form (similar to the powers of the imaginary unit i):

$$f^{(n)}(x) = \begin{cases} -\sin x,\ n = 4k-3 \\ -\cos x,\ n = 4k-2 \\ \sin x,\ n = 4k-1 \\ \cos x,\ n = 4k \end{cases},\ k \in \mathbb{Z}^+$$

$$R_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!}x^{n+1} \Rightarrow \left|f^{(n+1)}(t)\right| \le 1$$

$$0 < \frac{\left|f^{(n+1)}(t)\right|}{(n+1)!}\left|x^{n+1}\right| \le \frac{\left|x^{n+1}\right|}{(n+1)!}$$

$$\Rightarrow \lim_{n\to\infty}\frac{\left|x^{n+1}\right|}{(n+1)!} = 0 \Rightarrow \lim_{n\to\infty} R_n = 0$$

Hence $\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$ for all $x \in \mathbb{R}$.

2 **a** $f(x) = \sin^2 x = \sin x \times \sin x$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)\times\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$

$$= x^2 - \frac{2x^4}{3!} + \left(\frac{2}{5!} + \frac{1}{(3!)^2}\right)x^6 - \left(\frac{2}{7!} + \frac{2}{3!\times 5!}\right)x^8 + \ldots$$

$$= \frac{2x^2}{2!} - \frac{8x^4}{4!} + \frac{12+20}{6!}x^6 - \frac{16+112}{8!}x^8 + \ldots$$

$$= \frac{2x^2}{2!} - \frac{8x^4}{4!} + \frac{32}{6!}x^6 - \frac{128}{8!}x^8 + \ldots + (-1)^n\frac{2^{2n-1}x^{2n}}{(2n)!} + \ldots$$

$$= \sum_{n=1}^{\infty}(-1)^n\frac{2^{2n-1}x^{2n}}{(2n)!}$$

b We use the expension of $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ where instead of x we input $3x$

$$f(x) = e^{3x} = \frac{(3x)^0}{0!} + \frac{(3x)^1}{1!} + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \ldots$$

$$= 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \ldots + \frac{3^n x^n}{n!} + \ldots$$

c $f(x) = \ln(1-x) \Rightarrow f'(x) = -\frac{1}{1-x}$

$$\Rightarrow f''(x) = -\frac{1}{(1-x)^2} \Rightarrow f''(x) = -\frac{2}{(1-x)^3}$$

$$f^{(n)}(x) = -\frac{(n-1)!}{(1-x)^n},\ n \in \mathbb{Z}^+$$

$f(0) = \ln(1) = 0 \Rightarrow f'(0) = -1 \Rightarrow f''(0) = -1$

$\Rightarrow f''(0) = -2$

$f^{(n)}(0) = -(n-1)!,\ n \in \mathbb{Z}^+$

$$f(x) = \sum_{n=1}^{\infty} -\frac{(n-1)!}{n!}x^n = \sum_{n=1}^{\infty} -\frac{x^n}{n}$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^n}{n} - \ldots$$

d $f(x) = \tan x \Rightarrow f'(x) = (\cos x)^{-2}$

$\Rightarrow f''(x) = -2(\cos x)^{-3}\times(-\sin x) = 2(\cos x)^{-3}\sin x$

$\Rightarrow f^{(3)}(x) = -6(\cos x)^{-4}\times(-\sin x)\times\sin x$

$+ 2(\cos x)^{-3}\times(-\cos x)$

$= 2(\cos x)^{-4}(1 + 2(\sin x)^2)$

$\Rightarrow f^{(4)}(x) = -8(\cos x)^{-5}\times(-\sin x)(1 + 2(\sin x)^2)$

$+ 2(\cos x)^{-4}4\sin x\cos x$

$= 8\sin x(\cos x)^{-5}(2 + (\sin x)^2)$

$= 8(\cos x)^{-5}(2\sin x + (\sin x)^3)$

$\Rightarrow f^{(5)}(x) = -40(\cos x)^{-6}\times(-\sin x)(2\sin x + (\sin x)^3)$

$+ 8(\cos x)^{-5}(2\cos x + 3(\sin x)^2\cos x)$

$= 8(\cos x)^{-6}(10(\sin x)^2 + 5(\sin x)^4 + 2(\cos x)^2$

$+ 3(\sin x)^2(\cos x)^2)$

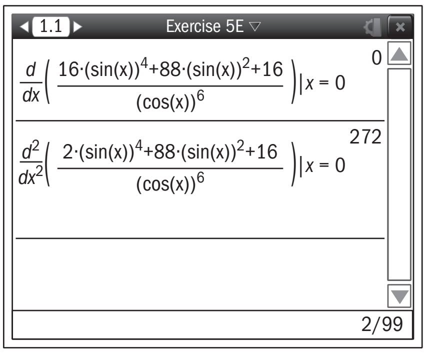
$= 8(\cos x)^{-6}(10(\sin x)^2 + 5(\sin x)^4 + 2(1-(\sin x)^2)$

$+ 3(\sin x)^2(1-(\sin x)^2))$

$$= \frac{16(\sin x)^4 + 88(\sin x)^2 + 16}{(\cos x)^6}$$

$f(0) = 0 \Rightarrow f'(0) = 1 \Rightarrow f''(0) = 0 \Rightarrow$

$f^{(3)}(0) = 2 \Rightarrow f^{(4)}(0) = 0 \Rightarrow f^{(5)}(0) = 16 \Rightarrow$



$$f(x) = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \frac{272x^7}{7!} + \ldots$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots$$

e The derivatives are best found using a computer algebra system.

$f(x) = \arctan(x^2);\ f(0) = 0$

$f'(x) = \dfrac{2x}{x^4+1};\ f'(0) = 0$

$f''(x) = \dfrac{-2(3x^4-1)}{(x^4+1)^2};\ f''(0) = 2$

$f'''(x) = \dfrac{8x^3(3x^4-5)}{(x^4+1)^3};\ f'''(0) = 0$

$f^{(4)}(x) = \dfrac{-24x^2(5x^8-22x^4+5)}{(x^4+1)^4};\ f^{(4)}(0) = 0$

$f^{(5)}(x) = \dfrac{48x(15x^{12}-135x^8+101x^4-5)}{(x^4+1)^5};$

$f^{(5)}(0) = 0$

$f^{(6)}(x) = \dfrac{-240(21x^{16}-336x^{12}+546x^8-120x^4+1)}{(x^4+1)^6};$

$f^{(6)}(0) = -240$

Hence,

$$f(x) = x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \ldots + (-1)^n\frac{x^{4n+2}}{2n+1} + \ldots$$

f $f(x) = e^x \times \sin x$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) \times \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$

$$= x + x^2 + \left(\frac{1}{2!} - \frac{1}{3!}\right)x^3 + \left(\frac{1}{3!} - \frac{1}{3!}\right)x^4 + \left(\frac{1}{5!} - \frac{1}{2!\times 3!} + \frac{1}{4!}\right)x^5 + \ldots$$

$$= x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \ldots$$

3 $f(x) = \sin^2 x = \dfrac{1}{2} - \dfrac{1}{2}\cos(2x)$

$$= \frac{1}{2} - \frac{1}{2}\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots\right)$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{2x^2}{2!} - \frac{2^3x^4}{4!}$$

$$+ \frac{2^5x^6}{6!} - \ldots = \sum_{n=1}^{\infty}(-1)^n\frac{2^{2n-1}x^{2n}}{(2n)!}$$

4 $f(x) = \dfrac{\sin x}{x} = \left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots\right) \times \dfrac{1}{x}$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots + (-1)^n\frac{x^{2n}}{(2n+1)!} + \ldots$$

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots\right)$$

$$= 1 - \frac{0^2}{3!} + \frac{0^4}{5!} - \frac{0^6}{7!} + \ldots = 1$$

5 We use the expansion of $e^x = \sum_{n=0}^{\infty}\dfrac{x^n}{n!}$ where instead of x we input ix and $-ix$ respectively.

1 $$e^{ix} = \frac{(ix)^0}{0!} + \frac{(ix)^1}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \ldots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{2!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \ldots$$

2 $$e^{ix} = \frac{(-ix)^0}{0!} + \frac{(-ix)^1}{1!} + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \frac{(-ix)^5}{5!} + \frac{(-ix)^6}{6!} + \frac{(-ix)^7}{7!} + \ldots$$

$$= 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{2!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \frac{ix^7}{7!} + \ldots$$

a If we subtract equation (2) from equation (1) we obtain the following:

$$e^{ix} - e^{-ix} = \left(1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \ldots\right)$$

$$- \left(1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \frac{ix^7}{7!} + \ldots\right)$$

$$= 2ix - \frac{2ix^3}{3!} + \frac{2ix^5}{5!} - \frac{2ix^7}{7!} + \ldots$$

$$= 2i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$

$$= 2i\sin x \Rightarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

b If we add equations (1) and (2) we obtain the following:

$$e^{ix} + e^{-ix} = \left(1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \ldots\right)$$

$$+ \left(1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \frac{ix^5}{5!} - \frac{x^6}{6!} + \frac{ix^7}{7!} + \ldots\right)$$

$$= 2 - \frac{2x^2}{2!} + \frac{2x^4}{4!} - \frac{2x^6}{6!} + \ldots$$

$$= 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right)$$

$$= 2\cos x \Rightarrow \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

6 We can solve this question by using the results from the previous question directly or using algebra.

Method I

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{i\theta^7}{7!} + \ldots$$

$$= \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right)}_{\sin\theta}$$

$$= \cos\theta + i\sin\theta$$

Method II

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},\ \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \Rightarrow \cos\theta + i\sin\theta$$

$$= \frac{e^{i\theta} + e^{-i\theta}}{2} + \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{2e^{i\theta}}{2} = e^{i\theta}$$

Exercise 5F

1 a $f(x) = \sqrt{1-x} = (1-x)^{\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-x)^n$, we are going to simplify the binomial coefficient

$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}\times\left(-\frac{1}{2}\right)\times\left(-\frac{3}{2}\right)\times\ldots\times\left(\frac{1}{2}-n+1\right)}{n!}$$

$$= (-1)^{n-1}\frac{1\times3\times5\times\ldots\times(2n-3)}{2^n\times n!}$$

$$= (-1)^{n-1}\frac{(2n)!}{2^n\times n!\times 2^n\times n!\times(2n-1)}$$

$$= (-1)^{n-1}\frac{(2n)!}{(2^n\times n!)^2(2n-1)},\ \text{so}$$

$$= 1 + \sum_{n=0}^{\infty}(-1)^{n-1}\frac{(2n)!}{(2^n\times n!)^2(2n-1)}(-1)^n x^n$$

$$= 1 - \sum_{n=0}^{\infty}\frac{(2n)!}{(2^n\times n!)^2(2n-1)}x^n$$

$|-x| < 1 \Rightarrow x \in]-1,1[$

b $f(x) = \frac{1}{(1+x)^3} = (1+x)^{-3} = \sum_{n=0}^{\infty}\binom{-3}{n}x^n$, we are going to simplify the binomial coefficient

$$\binom{-3}{n} = \frac{(-3)\times(-4)\times\ldots\times(-3-n+1)}{n!}$$

$$= (-1)^n\frac{3\times4\times\ldots\times(n+2)}{n!} = (-1)^n\frac{(n+2)!}{2\times n!}$$

$$= (-1)^n\frac{(n+1)(n+2)}{2} = (-1)^n\binom{n+2}{2},\ \text{so}$$

$$= \sum_{n=0}^{\infty}(-1)^n\binom{n+2}{2}x^n$$

$$= 1 - 3x + 6x^2 - 10x^3 + 21x^4 - 28x^5 + \ldots$$

$|x| < 1 \Rightarrow x \in]-1, 1[$

c $f(x) = \frac{1}{(1-4x^2)^2} = (1-4x^2)^{-2} = \sum_{n=0}^{\infty}\binom{-2}{n}(-4x^2)^n$,

we can simplify this binomial coefficient

$$\binom{-2}{n} = \frac{(-2)\times(-3)\times\ldots\times(-2-n+1)}{n!}$$

$$= (-1)^n\frac{2\times3\times\ldots\times(n+1)}{n!}$$

$$= (-1)^n\frac{(n+1)!}{n!} = (-1)^n(n+1),$$

$$\text{so} = \sum_{n=0}^{\infty}(-1)^n(n+1)(-4)^n x^{2n} = \sum_{n=0}^{\infty}4^n(n+1)x^{2n}$$

$$= 1 + 8x^2 + 48x^4 + 256x^6 + ..$$

$$\left|-4x^2\right| < 1 \Rightarrow x^2 < \frac{1}{4} \Rightarrow x \in \left]-\frac{1}{2},\frac{1}{2}\right[$$

d $f(x) = \frac{1}{\sqrt[4]{1+2x^3}} = \left(1+2x^3\right)^{-\frac{1}{4}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{4}}{n}\left(2x^3\right)^n$,

we are going to the binomial coefficient

$$\binom{-\frac{1}{4}}{n} = \frac{\left(-\frac{1}{4}\right)\times\left(-\frac{5}{4}\right)\times\left(-\frac{9}{4}\right)\times\ldots\times\left(-\frac{1}{4}-n+1\right)}{n!}$$

$$= (-1)^n\frac{1\times5\times9\times\ldots\times(4n-3)}{4^n\times n!},\ \text{so}$$

$$= 1 + \sum_{n=1}^{\infty}(-1)^n\frac{1\times5\times9\times\ldots\times(4n-3)}{4^n\times n!}(2x^3)^n$$

$$= 1 + \sum_{n=1}^{\infty}(-1)^n\frac{1\times5\times9\times\ldots\times(4n-3)}{2^n\times n!}x^{3n}$$

$$\left|2x^3\right| < 1 \Rightarrow \left|x^3\right| < \frac{1}{2} \Rightarrow |x| < \frac{1}{\sqrt[3]{2}}\ x \in \left]-\frac{1}{\sqrt[3]{2}},\frac{1}{\sqrt[3]{2}}\right[$$

2 $f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$

$$= (1-x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-x^2)^n$$

$$\binom{-\frac{1}{2}}{n} = \frac{\left(-\frac{1}{2}\right)\times\left(-\frac{3}{2}\right)\times\ldots\times\left(-\frac{1}{2}-n+1\right)}{n!}$$

$$= (-1)^n\frac{1\times3\times5\times\ldots\times(2n-1)}{2^n\times n!}$$

$$= (-1)^n\frac{(2n)!}{2^n\times n!\times2^n\times n!} = (-1)^n\frac{(2n)!}{(2^n\times n!)^2},\ \text{so}$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{(2^n\times n!)^2}(-1)^n x^{2n} = \sum_{n=0}^{\infty}\frac{(2n)!}{(2^n\times n!)^2}x^{2n}$$

$$f(x) = \int\sum_{n=0}^{\infty}\frac{(2n)!}{(2^n\times n!)^2}x^{2n}dx$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n \times n!)^2} \frac{x^{2n+1}}{2n+1} = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \dots$$

3 **a** $\sqrt{9+x} = 3\sqrt{1+\frac{x}{9}} = 3\left(1+\frac{x}{9}\right)^{\frac{1}{2}} = 3 \times \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}\left(\frac{x}{9}\right)^n$

$$= 3\left(1+\frac{1}{2}\times\frac{x}{9}-\frac{1}{8}\times\frac{x^2}{81}+\frac{1}{16}\times\frac{x^3}{729}-\dots\right)$$

$$= 1+\frac{1}{6}x-\frac{1}{216}x^2+\frac{1}{3888}x^3-\dots$$

$$\sqrt{9.1} = \sqrt{9+0.1} \approx 3+\frac{1}{6}\times 0.1-\frac{1}{216}\times 0.1^2 = 3.017$$

b $\frac{1}{(1+x)^4} = (1+x)^{-4} = \sum_{n=0}^{\infty}\binom{-4}{n}x^n$

$$= 1-4x+10x^2-20x^3+35x^4-\dots$$

$$\frac{1}{1.03^4} = \frac{1}{(1+0.03)^4}$$

$$\approx 1-4\times 0.03+10\times 0.03^2-20\times 0.03^3 = 0.8885$$

Exercises 5G

1 $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$

When $x=-1$,

$$e^{-1} = 1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}-\frac{1}{5040}+\dots.$$

The series alternates and the last term is less than 0.001

Error < $\frac{1}{5040}$, which is approximately 0.0002, hence

$$e^{-1} \approx 1-1+\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720} \approx 0.36806.$$

2 $\ln(1+x) = \int\frac{1}{1+x}dx = \int\sum(-1)^n x^n = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}+c.$

$|R_n| < u_{n+1} \Rightarrow \frac{0.5^{n+1}}{n+1} < 0.0001 \Rightarrow n=9$, and

$$\sum_{n=0}^{9}(-1)^n\frac{(0.5)^{n+1}}{n+1} = 0.405435$$

3 $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$

$$e^1 = 1+1+\frac{1}{2!}+\frac{1}{3!}+\dots \le 1+1+\frac{1}{2^2}+\frac{1}{2^3}+\dots$$

$$= 1+\frac{1}{1-\frac{1}{2}} = 3.$$ Hence, e < 3.

Since $|R_n(1)| = \frac{|f^{(n+1)}(c)|}{(n+1)!} = \frac{e^c}{(n+1)!}$, and e^x is a strictly increasing function,

$e^c < e < 1 \Rightarrow |R_n(1)| < \frac{3}{(n+1)!} < 0.00005 \Rightarrow n \ge 8.$

Hence, $e \approx 1+1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{8} = 2.7183.$

4 $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}+R_5(x);$

$R_5(x) = \frac{f^{(6)}(c)x^6}{6!} = \frac{-\sin c\times x^6}{6!}; c\in\left]-\frac{1}{3},\frac{1}{3}\right[.$

$f^{(6)}(c)$ is maximized when $c=\frac{1}{3}$.

Hence, the maximum error is

$$\frac{\left(\frac{1}{3}\right)^6}{6!}\times\sin\left(\frac{1}{3}\right) \approx 6.2337\times 10^{-7}.$$

5 $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^n}{n!}+\dots$

For $x=\frac{1}{3}$ $R_n\left(\frac{1}{3}\right) = \frac{f^{(n+1)}(c)\left(\frac{1}{3}\right)^{n+1}}{(n+1)!}, 0<c<\frac{1}{3}.$

Hence, $\left|R_n\left(\frac{1}{3}\right)\right| < \frac{e^{\frac{1}{3}}}{(n+1)!}\times\left(\frac{1}{3}\right)^{n+1}$. Since e < 27, $e^{\frac{1}{3}} < 3$,

and we need

$$\left|R_n\left(\frac{1}{3}\right)\right| < 0.00005 \Rightarrow \frac{1}{(n+1)!\times 3^n} < 0.00005.$$

Therefore, n = 5 since $\frac{1}{(5+1)!\times 3^5} \approx 0.000006 < 0.00005.$

Hence,

$$T_5\left(\frac{1}{3}\right) = 1+\frac{1}{3}+\frac{\left(\frac{1}{3}\right)^2}{2!}+\frac{\left(\frac{1}{3}\right)^3}{3!}+\frac{\left(\frac{1}{3}\right)^4}{4!}+\frac{\left(\frac{1}{3}\right)^5}{5!} \approx 1.3956.$$

6 $f(x) = x^{\frac{1}{3}};\ f'(x) = \frac{1}{3}x^{-\frac{2}{3}};$

$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}};\ f'''(x) = \frac{10}{27}x^{-\frac{8}{3}}.$

$f(8) = 2;\ f'(8) = \frac{1}{12};\ f''(8) = -\frac{1}{144}.$

Hence, $T_2(x) = f(8)+f'(8)(x-8)+\frac{f''(8)}{2!}(x-8)^2$

and $x^{\frac{1}{3}} \approx T_2(x)$.

$$|R_2(x)| = \frac{f'''(c)}{3!}(x-8)^3 = \frac{10}{27}c^{-\frac{8}{3}}\frac{(x-8)^3}{3!} = \frac{5(x-8)^3}{81c^{\frac{8}{3}}},$$

where 8 < c < x. Since $7 \le x \le 9$, $-1 \le (x-8)$, or $|x-8| \le 1$, and therefore

$|x-8|^3 \le 1.\ x > 7 \Rightarrow c^{\frac{8}{3}} > 7^{\frac{8}{3}} > 179.$

Hence, $|R_2(x)| = \frac{5|x-8|^3}{81c^{\frac{8}{3}}} < \frac{5\times 1}{81\times 179} < 0.0004$, i.e., the approximation is accurate to within 0.0004.

7 $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \Rightarrow \sin^2 x$

$$= (\sin x)(\sin x) = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + \ldots$$

$$\int_0^{\frac{\pi}{6}}\left(x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6\right)dx$$

$$= \left[\frac{x^3}{3} - \frac{1}{15}x^5 + \frac{2}{315}x^7\right]_0^{\frac{\pi}{6}} = 0.045294\ldots$$

The exact integral is, $\int_0^{\frac{\pi}{6}} \sin^2 x\,dx = \int_0^{\frac{\pi}{6}} \frac{1-2\cos x}{2}dx$

$$= \left[\frac{x}{2} - \frac{\sin\frac{\pi}{3}}{4}\right]_0^{\frac{\pi}{6}} = 0.045293\ldots$$

The truncated term $|R_4(x)| = \frac{8}{315}x^8$,

$\int_0^{\frac{\pi}{6}} \frac{x^8}{315}dx = \left[\frac{x^9}{2835}\right]_0^{\frac{\pi}{6}} = 0.000001$. The actual error is $0.0452940 - 0.045293 = 0.000001$ and $|R_4(x)| \le$ actual error.

8 $R_5\left(\frac{1}{2}\right) = \frac{f^{(6)}(c)}{6!}\times\left(\frac{1}{2}\right)^6, 0 < c < \frac{1}{2}$, and $\left|f^{(6)}(c)\right| < 1$.

$$|R_5(x)| \le \frac{1}{6!\times 26} \approx 0.0000217.$$

9 $R_4(0.2) = \frac{f^{(5)}(c)}{5!}\times(0.2)^5;\ 0 < c < 0.2$. Since $f^{(5)}(x) = e^x$, the maximum that $f^{(5)}(c)$ can be in this interval is $e^{0.2}$,

hence $|R_4(0.2)| \le \frac{e^{0.2}}{5!}\times(0.2)^5 \approx 0.0000326$.

10 $|R_n(1)| \le \left|\frac{\sin^{(n)}(1)}{(n+1)!}\right|$. Since all derivates of $\sin(x)$ are between -1 and 1, $|R_n(1)| \le \frac{1}{(n+1)!}$. By trial and error, n = 6, $\frac{1}{7!} = \frac{1}{5050} < 0.001$

The degree of the polynomial is therefore 5.

Exercise 5H

1 a $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

$$\lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right) - x}{x^3}$$

$$= -\frac{1}{3!}\lim_{x\to 0}\frac{x^2}{5!} + \frac{x^4}{7!} + \ldots = -\frac{1}{6}$$

b $$\lim_{x\to 0}\frac{\left(1 + x + \frac{x^2}{2!} + \ldots.\right) - \left(1 - x + \frac{x^2}{2!} - \ldots\right)}{x}$$

$$= \lim_{x\to 0}\frac{2x + \frac{2x^2}{2!} + \frac{2x^4}{4!}\ldots}{x} = 2\lim_{x\to 0}\frac{2x}{2!} + \frac{2x^3}{4!} + \ldots = 2$$

c $$\lim_{x\to 0}\frac{(2-2\cos x)^3}{x^6} = 8\lim_{x\to 0}\frac{(1-\cos x)^3}{x^6}$$

$$= 8\lim_{x\to 0}\frac{1 - 3\cos x + 3\cos^2 x - \cos^3 x}{x^6}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\cos^2 x = (\cos x)(\cos x)$$

$$= 1 - x^2 + \frac{x^4}{3} - \frac{2x^6}{45} + \ldots$$

$$\cos x = (\cos x)(\cos^2 x) = 1 - \frac{3x^2}{2} + \frac{7x^4}{8} - \frac{61x^6}{240} + \ldots$$

$$8\lim_{x\to 0}\frac{1 - 3\cos x + 3\cos^2 x - \cos^3 x}{x^6} = 8\lim_{x\to 0}\frac{1}{8}x^6 = 1$$

d $\sin^2 x = x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 + \ldots$

Hence, $\lim_{x\to 0}\frac{x^2 - \sin^2 x}{x^2\sin^2 x} = \lim_{x\to 0}\frac{\frac{1}{3}x^4 - \frac{2}{45}x^6 + \ldots}{x^4 - \frac{x^6}{3} + \ldots} = \frac{1}{3}$.

e $$\lim_{x\to 0}\frac{(\sin 3x - 3x)^2}{(\cos 5x - 1)^3}$$

$$= \lim_{x\to 0}\frac{\sin^2 3x + 3x\sin 3x + 9x^2}{\cos^3 5x - 3\cos^2 5x + 3\cos 5x - 1} = -\frac{162}{15625}$$

Numerator: $\left[(3x)^2 - \frac{(3x)^4}{3} + \frac{2(3x)^6}{45} - \ldots\right]$

$$-3x\left[3x - \frac{(3x)^3}{6} + \frac{(3x)^5}{120} - \ldots\right] + (3x)^2$$

Denominator:

$$\left[1 - \frac{3(5x)^2}{2} + \frac{7(5x)^4}{8} - \ldots\right] - 3\left[1 - (5x)^2 + \frac{(5x)^4}{3} - \ldots\right]$$

$$+3\left[1 - \frac{(5x)^2}{2} + \frac{(5x)^4}{24} - \ldots\right] - 1$$

2 For c = 0, $y = y(0) + y'(0) + \frac{y''(0)}{2!} + \frac{y'''(0)}{3!} + \ldots$

Therefore, $y(0) = 0$;

$y' = y^2 - x;\ y'(0) = 1$

$y'' = 2yy' - 1;\ y''(0) = 1$

$y''' = 2yy'' + 2(y')^2;\ y'''(0) = 4$

$y^{(4)} = 2yy''' + 6y'y'';\ y^{(4)}(0) = 14$

$y^{(5)} = 2yy^{(4)} + 8y'y''' + 6(y'')^2;\ y^{(5)}(0) = 66$

Hence, $y = 1 + x + \frac{1}{2}x^2 + \frac{4}{3!}x^3 + \frac{14}{4!}x^4 + \frac{66}{5!}x^5 + \ldots$

$y(0.2) = 1.2264$ (4d.p.)

3 $y' = 3x^2y$;

$y'' = 3x^2y' + 6xy = 3x^2(3x^2y) + 6xy = 6xy + 9x^4y$

$y''' = 6y + 6xy' + 36x^3y + 9x^4y'$

$= 6y + 6x(3x^2y) + 36x^3y + 9x^4(3x^2y)$

$= 6y + 54x^3y + 27x^6y$

For (1,1), $y' = 3;\ y'' = 15;\ y''' = 87$

Therefore, using the Taylor series expansion,

$y = 1 + (x-1)(3) + \frac{(x-1)^2}{2!}(15) + \frac{(x-1)^3}{3!}(87)$

$= 1 + 3(x-1) + 7.5(x-1)^2 + 14.5(x-1)^3$

4 In this example we will purposely take a different approach to show you another way of solving this type of problem.

Let $y = \sum c_n x^n;\ y' = \sum nc_n x^{n-1}$.

Since $x^2y' = y - x - 1$,

$x^2y' = x^2\sum_{n=1}^{\infty} nc_n x^{n-1} = -1 - x + \sum_{n=1}^{\infty} c_n x^n$, so that

$\sum_{n=1}^{\infty} nc_n x^{n+1} = -1 - x + \sum_{n=0}^{\infty} c_n x^n$. Moving the summation index on the left by −1, i.e., replace n = 1 with n = 2 and n with (n − 1) we obtain

$\sum_{n=2}^{\infty}(n-1)\,c_{n-1}x^n = -1 - x + c_0 + c_1x + \sum_{n=2}^{\infty} c_n x^n$.

$c_0 = 1 \times c_1 = 1!\,;\ c_3 = 2 \times c_2 = 2!;$

$c_4 = 3 \times c_3 = 3!;\ldots;\ c_n = (n-1)!,\ n \geq 2$.

Hence, $y(x) = 1 + x + \sum_{n=2}^{\infty}(n-1)!x^n$.

The radius of convergence is $\lim_{n\to\infty}\frac{(n-1)!}{n!} = \lim_{n\to\infty}\frac{1}{n} = 0$.

Therefore the series converges only for $x = 0$. Hence, the differential equation does not have a convergent power series solution.

Review Exercise

1 Using the ratio test,

$\lim_{n\to\infty}\left(\frac{(2n)!x^{n+1}}{n!(n+1)!} \times \frac{n!(n-1)!}{(2n-2)!x^n}\right)$

$= \lim_{n\to\infty}\left(\frac{2n(2n-1)}{n(n+1)}\right)x = 4x$. The series is convergent

if $|4x| < 1$, or $x \in \left]-\frac{1}{4}, \frac{1}{4}\right[$, hence $R = \frac{1}{4}$.

2 **a** **i** $f'(x) = -e^{-x}\cos 2x - 2e^{-x}\sin 2x$

$f''(x) = 4e^{-x}\sin 2x - 3e^{-x}\cos 2x$

$f''(x) + 2f'(x) + 5(x)$

$= 4e^{-x}\sin 2x - 3e^{-x}\cos 2x - 2e^{-x}\cos 2x - 4e^{-x}\sin 2x + 5e^{-x}\cos 2x$.

Hence, $f''(x) + 2f'(x) + 5f(x) = 0$.

ii Differentiating n times and putting $x = 0$,

$f^{(n+2)}(0) + 2f^{(n+1)}(0) + 5f^{(n)}(0) = 0$.

b $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \ldots$ and

$\cos 2x = 1 - 2x^2 + \frac{2x^4}{3} + \ldots$

Multiplying these,

$e^{-x}\cos 2x = 1 - x - \frac{3}{2}x^2 + \frac{11}{6}x^3 - \frac{7}{24}x^4 + \ldots$.

3 **a** f is a continuous and positive decreasing function, hence we can use the integral test.

$\int_1^{\infty}\frac{x}{e^{x^2}}dx = \lim_{b\to\infty}\int_1^b xe^{-x^2}dx = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^b = \frac{1}{2e}$.

Since the integral converges, the series converges.

b $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$ and

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

Hence, $e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots$

c $e^{x^2}\sin x = \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \ldots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$

$= x + \frac{5}{6}x^3 + \frac{41}{120}x^5 + \ldots$

d $\frac{e^{x^2}\sin x - x}{x^3} = \frac{5}{6} + \frac{41}{120}x^2$.

Hence, $\lim_{x\to 0}\left(\frac{e^{x^2}\sin x - x}{x^3}\right) = \frac{5}{6}$.

4 a i $a_n = \dfrac{f^{(n)}(0)}{n!}$ or $f^{(n)}(0) = n!a_n$.

When $x = 0$, $(n+2)!a_{n+2} - n^2 \times n!a_n = 0$.

Hence, $(n+1)(n+2)a_{n+2} = n^2 a_n$, $n \ge 1$.

ii $a_3 = \dfrac{1^2}{2\times 3}$; $a_5 = \dfrac{1^2}{2\times 3} \times \dfrac{3^2}{4\times 5}$.

Hence, $a_n = \dfrac{1^2 \times 3^2 \ldots (n-2)^2}{n!}$, $n \in$ Odds, $n \ge 3$.

b Using the ratio test, for odd n,

$\lim\limits_{n\to\infty} \dfrac{n^2}{(n+1)(n+2)} x^2 = x^2$. Hence, the series is

convergent for $|x| < 1$, and $R = 1$.

c $\arcsin\left(\dfrac{1}{2}\right) = \dfrac{\pi}{6} \approx 1 \times \dfrac{1}{2} + \dfrac{1}{6} \times \left(\dfrac{1}{2}\right)^3 + \dfrac{3}{40}\left(\dfrac{1}{2}\right)^5$.

≈ 3.139.

5 $\lim\limits_{n\to\infty} \left| \dfrac{(x-3)^{n+1}}{2^{n+1}\sqrt{n+1}} \times \dfrac{2^n\sqrt{n}}{(x-3)^n} \right| = \dfrac{|x-3|}{2}$. The series will

converge for $\dfrac{|x-3|}{2} < 1 \Rightarrow |x-3| < 2 \Rightarrow 1 < x < 5$.

Checking the endpoints, $x = 1$, $\sum\limits_{n=1}^{\infty} \dfrac{(-2)^n}{2^n\sqrt{n}} = \sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n}}$.

This converges by the alternating series test, since the terms decrease to zero. For $x = 5$,

$\sum\limits_{n=1}^{\infty} \dfrac{(2)^n}{2^n\sqrt{n}} = \sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$, which diverges by the p-series test.

Hence, the interval of convergence is [1,5]

6 Maclaurin series for $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$, $|R_n(x)| \le \dfrac{M}{(n+1)!} x^{n+1}$.

$|R_n(0.2)| \le \dfrac{e^{0.2}}{(n+1)!} 0.2^{n+1} < 0.0005$.

$e < 3 \Rightarrow e^{0.2} < e^{0.5} < 2$

Then, $|R_n(0.2)| \le \dfrac{2}{(n+1)!} 0.2^{n+1} < 0.0005 \Rightarrow n = 3$

Hence, $e^{0.2} \approx 1 + 0.2 + \dfrac{0.2^2}{2!} + \dfrac{0.2^3}{3!} = 1.221$

7 $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \ldots$.

$\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots + c$

Since $\ln(1) = 0$, $c = 0$.

$\ln(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \ldots$,

hence $\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) = \dfrac{1}{2}(\ln(1+x) - \ln(1-x))$

$$= \frac{1}{2}\left(\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right)\right)$$

$$= \frac{1}{2}\left(2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \ldots\right)$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots$$

8 $f(x) = (1+x)^{\frac{1}{3}} \Rightarrow f'(x) = \dfrac{1}{3}(1+x)^{-\frac{2}{3}}$;

$f''(x) = -\dfrac{2}{9}(1+x)^{\frac{5}{3}}$; $f'''(x) = \dfrac{10}{27}(1+x)^{\frac{8}{3}}$.

Hence, $f(x) \approx \sum\limits_{i=0}^{3} \dfrac{f^{(i)}(0)}{i!} x^i = 1 + \dfrac{1}{3}x - \dfrac{1}{9}x^2 + \dfrac{5}{81}x^3$;

$f(0.2) \approx 1.06272$.

9 $3^\circ = \dfrac{\pi}{180} \times 3 = \dfrac{\pi}{60}$.

$f(x) = f(0) + xf'(0) + \dfrac{x^2}{2!} f''(0) + \ldots + R_n$

$\Rightarrow \sin x = 0 + x - \dfrac{x^3}{3!} + \ldots + \dfrac{x^{n+1}}{(n+1)!} f^{(n+1)}(c)$

$f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$, then $f^{(n+1)}(c) = 1$.

$|R_n| \le \dfrac{\left|\frac{\pi}{60}\right|^{n+1}}{(n+1)!} \le 0.000005 \Rightarrow n = 3$.

$\sin 3^\circ \approx \dfrac{\pi}{60} - \dfrac{\left(\frac{\pi}{60}\right)^3}{3!} \approx 0.05234$.

10 a $f^{(n)}(x) = \dfrac{e^x + (-1)^n e^{-x}}{2}$

b Coefficient of $x^n = \dfrac{f^{(n)}(0)}{n!} = \dfrac{1+(-1)^n}{2n!}$

$\Rightarrow f(x) = 1 + \dfrac{x^2}{2} + \dfrac{x^4}{24} + \ldots.$

c $f(\frac{1}{2}) = 1 + \dfrac{1}{8} + \dfrac{1}{16 \times 24} = \dfrac{433}{384}$. The Lagrange error term $\dfrac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} = \dfrac{f^{(5)}(c)}{120} \times \left(\dfrac{1}{2}\right)^5$.

From the GDC, $f^{(5)}(c)$ is an increasing function, hence the expression is maximized when $x = 0.5$. Therefore the upper bound $= \dfrac{(e^{0.5} - e^{-0.5})}{2 \times 120} \times \left(\dfrac{1}{2}\right)^5 = 0.000136$.

11 a $f(x) = \ln(1 + \sin x)$. Hence, $f'(x) = \dfrac{\cos x}{1 + \sin x}$;

$f''(x) = \dfrac{-\sin x(1 + \sin x) - \cos^2 x}{(1 + \sin x)^2}$

$= \dfrac{-\sin x - 1}{(1 + \sin x)^2} = \dfrac{-1}{1 + \sin x}$.

b $f'''(x) = \dfrac{\cos x}{(1 + \sin x)^2}; f^{(4)}(x)$

$= \dfrac{-\sin x(1 + \sin x)^2 - 2(1 + \sin x)\cos^2 x}{(1 + \sin x)^4}$.

$f(0) = 0; f'(0) = 1; f''(0) = -1;$

$f'''(0) = 1; f^{(4)}(0) = -2.$

Therefore, $\ln(1 + \sin x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{6} - \dfrac{x^4}{12} + \ldots.$

c $\ln(1 - \sin x) = \ln(1 + \sin(-x))$

$= -x - \dfrac{x^2}{2} - \dfrac{x^3}{6} - \dfrac{x^4}{12} + \ldots$

d Adding,

$\ln(1 - \sin^2 x) = \ln\cos^2 x = -x^2 - \dfrac{x^4}{6} + \ldots$

$\ln\cos x = -\dfrac{x^2}{2} - \dfrac{x^4}{12} + \ldots$

$\ln\sec x = \dfrac{x^2}{2} + \dfrac{x^4}{12} + \ldots$

e $\dfrac{\ln\sec x}{x\sqrt{x}} = \dfrac{\sqrt{x}}{2} + \dfrac{x^2\sqrt{x}}{12} + \ldots$

Hence, $\lim_{x\to 0}\dfrac{\ln\sec x}{x\sqrt{x}} = 0$.

12 $y(0) = 0$

$y' = y^2 + 1;\ y'(0) = 1$

$y'' = 2yy';\ y''(0) = 0$

$y''' = 2\left[(y')^2 + yy''\right]; y'''(0) = 2;\ y^{(5)}(0) = 16$

$y^{(4)} = 2\left[2y'y'' + y'y'' + yy'''\right]$

$= 2\left[3y'y'' + yy'''\right];\ y^{(4)}(0) = 0$

$y^{(5)} = 2\left[3\left((y'')^2 + y'y'''\right) + y'y''' + yy^{(4)}\right]$

$= 2\left[3(y'')^2 + 4y'y'' + yy^{(4)}\right]$

$y = x + \dfrac{2x^3}{3!} + \dfrac{16x^5}{5!} + \ldots = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \ldots,$
which is the Maclaurin expansion of

$y = \tan(x)$.

13 a $y' = y\tan x + \cos x,\ f'(0) = 1$

$y'' = y\sec^2 x + y'\tan x - \sin x$

$\Rightarrow f''(0) = -\dfrac{\pi}{2}$

$\Rightarrow y = -\dfrac{\pi}{2} + x - \dfrac{\pi x^2}{4}$

b integrating factor

$= e^{\int -\tan x dx} = \cos x$

$\Rightarrow y\cos x = \int \cos^2 x dx = \dfrac{1}{2}\int(1 + \cos 2x)dx$

$\Rightarrow y\cos x = \dfrac{x}{2} + \dfrac{\sin 2x}{4} + c$

when

$x = \pi, y = 0 \Rightarrow c = -\dfrac{\pi}{2}$

$\Rightarrow y\cos x = \dfrac{x}{2} + \dfrac{\sin 2x}{4} - \dfrac{\pi}{2}$

$\Rightarrow y = \sec x\left(\dfrac{x}{2} + \dfrac{\sin 2x}{4} - \dfrac{\pi}{2}\right)$

14 $\sqrt{1+x-} = (1+x)^{\frac{1}{2}}$

$$= 1 + \left(\frac{1}{2}\right)x + \left(\frac{1}{2}\right)\cdot\left(-\frac{1}{2}\right)\frac{x^2}{2!} + \left(\frac{1}{2}\right)\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\frac{x^3}{3!} + \ldots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots$$

$$I =]-1,1[\quad \frac{1}{\sqrt{1-2x}} = (1-2x)^{-\frac{1}{2}}$$

$$= 1 + \left(-\frac{1}{2}\right)(-2x) + \left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\frac{(-2x)^2}{2!}$$

$$+ \left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{(-2x)^3}{3!} + \ldots$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + \ldots$$

$$I = \left]-\frac{1}{2}, \frac{1}{2}\right[$$

$$(1+x)^{\frac{1}{2}}(1-2x)^{-\frac{1}{2}}$$

$$= \left(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots\right)\left(1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + \ldots\right)$$

$$= 1 + \frac{3}{2}x + \frac{15}{8}x^2 + \frac{51}{16}x^3 + \ldots$$

$$I = \left]-\frac{1}{2}, \frac{1}{2}\right[$$

15 $e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$

$$e^{e^x - 1} = 1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6}\right) + \frac{\left(x + \frac{x^2}{2} + \frac{x^3}{6}\right)^2}{2}$$

$$+ \frac{\left(x + \frac{x^2}{2} + \frac{x^3}{6}\right)^3}{6} + \ldots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

$$= 1 + x + x^2 + \frac{5}{6}x^3 + \ldots$$

Using l'hopital's rule,

$$\lim_{x\to 0}\frac{e^{(e^x-1)} - 1}{e^{(e^x-1)} - 1 - 1} = \lim_{x\to 0}\frac{e^{(e^x-1)} - 1}{e^{(e^x+x-1)} - 1}$$

$$= \lim_{x\to 0}\frac{e^{(e^x+x-1)} - 1}{e^{(e^x+x-1)} \times (e^x + 1)} = \frac{1}{2}$$

16 $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots \Rightarrow \pi = 4\cdot\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{4}{2n-1}.$$

$$\frac{4}{2n-1} < 5\times 10^{-5} \Rightarrow n \geq 4{,}001$$

17 $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots$

$$e^{-\frac{x^2}{2}} = 1 + \left(-\frac{x^2}{2}\right) + \frac{\left(-\frac{x^2}{2}\right)^2}{2!} + \ldots + \frac{\left(-\frac{x^2}{2}\right)^n}{n!} + \ldots$$

$$= \sum_0^{\infty}(-1)^n\frac{x^{2n}}{2^n n!}$$

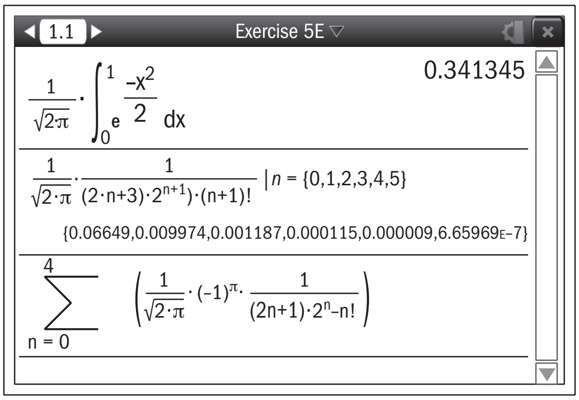
Hence, $P(0 < x < 1) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\int_0^1 \frac{(-1)^n x^{2n}}{2^n n!}dx$

$$= \frac{1}{2\pi}\sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+1}}{(2n+1)2^n n!}\right]_0^1.$$

Since this is an alternating series,

$$u_{n+1} = \frac{1}{(2n+3)2^{n+1}(n+1)!} < 0.0001.$$

From the GDC:



When n = 4, we have the appropriate approximation, i.e., 0.341354.